5. A step beyond linearization: velocity analysis
**Partially linearized seismic inverse problem** ("velocity analysis"): given observed seismic data $S^{\text{obs}}$, find smooth velocity $v \in \mathcal{E}(X), X \subset \mathbb{R}^3$ oscillatory reflectivity $r \in \mathcal{E}'(X)$ so that

$$F[v] r \simeq S^{\text{obs}}$$

Acoustic partially linearized model: acoustic potential field $u$ and its perturbation $\delta u$ solve

$$\left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) u = \delta(t) \delta(x - x_s), \quad \left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta u = 2r \nabla^2 u$$

plus suitable bdry and initial conditions.

$$F[v] r = \left. \frac{\partial \delta u}{\partial t} \right|_Y$$

*data acquisition manifold* $Y = \{(x_r, t; x_s)\} \subset \mathbb{R}^7$, dimn $Y \leq 5$ (many idealizations here!).
\[ F[v] : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y) \] is a linear map (FIO of order 1), but dependence on \( v \) is quite nonlinear, so this inverse problem is nonlinear.

**Agenda:**

- reformulation of inverse problem via *extensions*
- “standard processing” extension and standard VA
- the surface oriented extension and standard MVA
- the \( \Psi DO \) property and why it’s important
- global failure of the \( \Psi DO \) property for the SOE
- Claerbout’s depth oriented extension has the \( \Psi DO \) property
Extension of $F[v]$: manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

$$
\begin{array}{ccc}
\mathcal{E}'(\bar{X}) & \to & \mathcal{D}'(Y) \\
\chi & \uparrow & \uparrow \text{id} \\
\mathcal{E}'(X) & \to & \mathcal{D}'(Y) \\
F[v] & \text{commutes.}
\end{array}
$$

Invertible extension: $\bar{F}[v]$ has a right parametrix $\bar{G}[v]$, i.e. $I - \bar{F}[v]\bar{G}[v]$ is smoothing. [The trivial extension - $\bar{X} = X, \bar{F} = F$ - is virtually never invertible.] Also $\chi$ has a left inverse $\eta$.

Reformulation of inverse problem: given $S^{\text{obs}}$, find $v$ so that $\bar{G}[v]S^{\text{obs}} \in \mathcal{R}(\chi)$ (implicitly determines $r$ also!).
Example 1: Standard VA extension. Treat each CMP as if it were the result of an experiment performed over a layered medium, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(x_m, z), r = r(x_m, z)$.

$$F[v]R(x_m, h, t) \simeq A(x_m, h, z(x_m, h, t))R(x_m, z(x_m, h, t))$$

Here $z(x_m, h, t)$ is the inverse of the 2-way traveltime

$$t(x_m, h, z) = 2\tau(x_m + (h, 0, z), x_m)_{v=v(x_m, z)}$$

computed with the layered velocity $v(x_m, z)$, i.e.

$z(x_m, h, t(x_m, h, z')) = z'$.

$R$ is (yet another version of) “reflectivity”

$$R(x_m, z) = \frac{1}{2} \frac{dr}{dz}(x_m, z)$$
That is, $F[v]$ is a change of variable followed by multiplication by a smooth function. NB: industry standard practice is to use vertical traveltime $t_0$ instead of $z$ for depth variable.

Can write this as $F[v] = \bar{F} S^*$, where $\bar{F}[v] = N[v]^{-1} M[v]$ has right parametrix $\bar{G}[v] = M[v] N[v]$:

$N[v] = \textbf{NMO operator } N[v] d(x_m, h, z) = d(x_m, h, t(x_m, h, z))$

$M[v] = \text{multiplication by } A$

$S = \textbf{stacking operator}$

$S f(x_m, z) = \int dh f(x_m, h, z), \ S^* r(x_m, h, z) = r(x, z)$
Identify as extension: $\bar{F}[v], \bar{G}[v]$ as above, $X = \{x_m, z\}, H = \{h\}, \bar{X} = X \times H, \chi = S^*, \eta = S$ - the invertible extension properties are clear.

Standard names for the Standard VA extension objects: $\bar{F}[v]$ = “inverse NMO”, $\bar{G}[v]$ = “NMO” [often the multiplication op $M[v]$ is neglected]; $\eta$ = “stack”, $\chi$ = “spread”

**How this is used for velocity analysis:** Look for $v$ that makes $\bar{G}[v]d \in \mathcal{R}(\chi)$

So what is $\mathcal{R}(\chi)$? $\chi[r](x_m, z, h) = r(x_m, z)$ Anything in range of $\chi$ is independent of $h$. Practical issues ⇒ replace “independent of” with “smooth in”.
Inverse problem reduced to: adjust \( v \) to make \( \bar{G}[v]d^{\text{obs}} \) smooth in \( h \), i.e. flat in \( z, h \) display for each \( x_m \) (\textit{NMO-corrected CMP}).

Replace \( z \) with \( t_0 \), \( v \) with \( v_{\text{RMS}} \) em localizes computation: reflection through \( x_m, t_0, 0 \) \textit{flattened} by adjusting \( v_{\text{RMS}}(x_m, t_0) \Rightarrow 1D \) search, do by visual inspection.

Various aids - NMO corrected CMP gathers, velocity spectra, etc.

See: Claerbout: \textit{Imaging the Earth’s Interior}

WWS: MGSS 2000 notes
Left: part of survey ($S^{\text{obs}}$) from North Sea (thanks: Shell Research), lightly preprocessed.

Right: restriction of $\bar{G}[v]S^{\text{obs}}$ to $x_m = \text{const}$ (function of depth, offset): shows rel. sm’ness in $h$ (offset) for properly chosen $v$. 
This only works where Earth is “nearly layered”. Where this fails, go to **Example 2: Surface oriented or standard MVA extension**.

Shot version: $\Sigma_s = \{\text{set of shot locations}\}$, $\bar{X} = X \times \Sigma_s$, $\chi[r](x, x_s) = r(x)$.

$$
\bar{F}[v]\bar{r}(x_r, t, x_s) = \frac{\partial^2}{\partial t^2} \int dx \bar{r}(x, x_s) \int ds \, G(x_r, t - s; x)G(x_s, s; x)
$$

Offset version (preferred because it minimizes truncation artifacts): $\Sigma_h = \{\text{set of half-offsets in data}\}$, $\bar{X} = X \times \Sigma_h$, $\chi[r](x, h) = r(x)$.

$$
\bar{F}[v]\bar{r}(x_s, t, h) = \frac{\partial^2}{\partial t^2} \int dx \bar{r}(x, h) \int ds \, G(x_s + h, t - s; x)G(x_s, s; x)
$$

[Parametrize data with source location $x_s$, time $t$, offset $h$.] **NB:** note that both versions are “block diagonal” - family of operators (FIOs - ten Kroode lectures) parametrized by $x_s$ or $h$. 


Properties of surface oriented extension (Beylkin (1985), Rakesh (1988)): if \( \|v\|_{C^2(X)} \) “not too big”, then

- \( \tilde{F} \) has the \( \Psi DO \) property: \( \tilde{F}\tilde{F}^* \) is \( \Psi DO \)

- singularities of \( \tilde{F}\tilde{F}^*d \subset \) singularities of \( d \)

- straightforward construction of right parametrix \( \tilde{G} = \tilde{F}^*Q \), \( Q = \Psi DO \), also as generalized Radon Transform - explicitly computable.

Range of \( \chi \) (offset version): \( \tilde{r}(x, h) \) independent of \( h \) \( \Rightarrow \) “semblance principle”: find \( v \) so that \( \tilde{G}[v]d^{obs} \) is independent of \( h \). Practical limitations \( \Rightarrow \) replace “independent of \( h \)” by “smooth in \( h \)”.

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Application of these ideas = industrial practice of migration velocity analysis.

Idea: twiddle $v$ until $\tilde{G}[v]d^{obs}$ is smooth in $h$.

Since it is hard to inspect $\tilde{G}[v]d^{obs}(x, y, z, h)$, pull out subset for constant $x, y = \text{common image gather} (\text{“CIG”}):$ display function of $z, h$ for fixed $x, y$. These play same role as NMO corrected CMP gathers in layered case.

Try to adjust $v$ so that selected CIGs are flat - just as in Standard VA. This is much harder, as there is no RMS velocity trick to localize the computation - each CIG depends globally on $v$.

Description, some examples: Yilmaz, *Seismic Data Processing*. 
Nolan (1997): big trouble! In general, standard extension does \textbf{not} have the $\Psi\Do$ property. Geometric optics analysis: for $\|v\|_{C^2(X)}$ “large”, multiple rays connect source, receiver to reflecting points in $X$; block diagonal structure of $\bar{F}[v] \Rightarrow$ info necessary to distinguish multiple rays is \textit{projected out}.

Example (Stolk & WWS, 2001): Gaussian lens over flat reflector at depth $z$ ($r(x) = \delta(x_1 - z)$, $x_1 =$ depth).
**Left:** Const. $h$ slice of $\bar{G}d_{\text{obs}}$: several refl. points corresponding to same singularity in $d_{\text{obs}}$.

**Right:** CIG (const. $x, y$ slice) of $\bar{G}d_{\text{obs}}$: not smooth in $h$!
Standard MVA extension only works when Earth has simple ray geometry. When this fails, go to

**Example 3:** Claerbout’s depth oriented extension.

$\Sigma_d = \text{somewhat arbitrary set of vectors near } 0 \text{ ("offsets")}, \quad \bar{X} = X \times \Sigma_d, \quad \chi[r](x, h) = r(x)\delta(h), \quad \eta[\bar{r}](x) = \bar{r}(x, 0)$

$$\bar{F}[v]\bar{r}(x_s, t, x_r) = \frac{\partial^2}{\partial t^2} \int dx \int_{\Sigma_d} dh \bar{r}(x, h) \int ds G(x_s, t-s; x+2h)G(x_r, s; x)$$

$$= \frac{\partial^2}{\partial t^2} \int dx \int_{x+2\Sigma_d} dy \bar{r}(x, y-x) \int ds G(x_s, t-s; y)G(x_r, s; x)$$

**NB:** in this formulation, there appears to be too many model parameters.
Computationally economical: for each $x_s$ solve

$$\bar{F}[v]\bar{r}(x_r, t; x_s) = u(x, t; x_s)|_{x=x_r}$$

where

$$\left(\frac{1}{v(x)^2} \frac{\partial^2}{\partial t^2} - \nabla^2_x\right) u(x, t; x_s) = \int_{x+2\Sigma_d} dy \bar{r}(x, y) G(y, t; x_s)$$

$$\left(\frac{1}{v(y)^2} \frac{\partial^2}{\partial t^2} - \nabla^2_y\right) G(y, t; x_s) = \delta(t)\delta(x_s - y)$$

Finite difference scheme: form RHS for eqn 1, step $u$ forward in $t$, step $G$ forward in $t$. 
Computing $\bar{G}[v]$: instead of parametrix, be satisfied with adjoint.

Reverse time adjoint computation - specify adjoint field as in standard reverse time prestack migration:

$$\left(\frac{1}{v(x)^2} \frac{\partial^2}{\partial t^2} - \nabla_x^2\right) w(x, t; x_s) = \int dx_r d(x_r, t; x_s) \delta(x - x_r)$$

with $w(x, t; x_s) = 0, t >> 0$.

Then

$$\bar{F}[v]^* d(x, h) = \int dx_s \int dt G(x + 2h, t; x_s) w(x, t; x_s)$$

i.e. exactly the same computation as for reverse time prestack, except that crosscorrelation occurs at an offset $2h$. 
What should be the character of the image when the velocity is correct?

Hint: for simulation of seismograms, the input reflectivity had the form $r(x)\delta(h)$.

Therefore guess that when velocity is correct, *image is concentrated near $h = 0$.*

Examples: 2D finite difference implementation of reverse time method. Correct velocity $\equiv 1$. Input reflectivity used to generate synthetic data: random! For output reflectivity (image of $\bar{F}[v]^*$), constrain offset to be horizontal: $\bar{r}(x, h) = \tilde{r}(x, h_1)\delta(h_3)$. Display CIGs (i.e. $x_1 = \text{const.}$ slices).
Stolk and deHoop, 2001: Claerbout extension has the $\Psi$DO property, at least when restricted to $\tilde{r}$ of the form $\tilde{r}(x, h) = R(x, h_1, h_2)\delta(h_3)$, and under DSR assumption.

Sketch of proof (after Rakesh, 1988):

This will follow from injectivity of wavefront or canonical relation $C_{\tilde{F}} \subset T^*(\tilde{X}) - \{0\} \times T^*(Y) - \{0\}$ which describes singularity mapping properties of $\tilde{F}$:

$$(x, h, \xi, \nu, y, \eta) \in C_{F_\delta}[v] \Leftrightarrow$$

for some $u \in \mathcal{E}'(\tilde{X})$, $(x, h, \xi, \nu) \in WF(u)$, and $(y, \eta) \in WF(\tilde{F}u)$
Characterization of $C_{\overline{F}}$:

$$((x, h, \xi, \nu), (x_s, t, x_r, \xi_s, \tau, \xi_r)) \in C_{\overline{F}}[v] \subset T^*(\overline{X}) - \{0\} \times T^*(Y) - \{0\}$$

$\Leftrightarrow$ there are rays of geometric optics $(X_s, \Xi_s)$, $(X_r, \Xi_r)$ and times $t_s, t_r$ so that

$$\Pi(X_s(0), t, X_r(0), \Xi_s(0), \tau, \Xi_r(0)) = (x_s, t, x_r, \xi_s, \tau, \xi_r),$$

$$X_s(t_s) = x, X_r(t_r) = x + 2h, t_s + t_r = t,$$

$$\Xi_s(t_s) + \Xi_r(t_r)\|\xi, \Xi_s(t_s) - \Xi_r(t_r)\|\nu$$
$t_s + t = t_s' + t_s'$.  

$X_s \xi_s \quad X(t_s) \Xi(t_s)$  

$X(t_s') \Xi(t_s')$  

$X(t_s') \Xi(t_s')$  

$X(t_s) \Xi(t_s')$  

$X(t_s') \Xi(t_s')$
Proof: uses wave equations for $u, G$ and

- Gabor calculus: computes wave front sets of products, pull-backs, integrals, etc. See Duistermaat, Ch. 1.

- Propagation of Singularities Theorem

and that’s all! [No integral representations, phase functions,...]
Note intrinsic ambiguity: if you have a ray pair, move times $t_s, t_r$ resp. $t'_s, t'_r$, for which $t_s + t_r = t'_s + t'_r = t$ then you can construct two points $(x, h, \xi, \nu), (x', h', \xi', \nu')$ which are candidates for membership in $WF(\bar{r})$ and which satisfy the above relations with the same point in the cotangent bundle of $T^*(Y)$.

No wonder - there are too many model parameters!

Stolk and deHoop fix this ambiguity by imposing two constraints:

- DSR assumption: all rays carrying significant reflected energy (source or receiver) are upcoming.
- Restrict $\bar{F}$ to the domain $\mathcal{Z} \subset \mathcal{E}'(\bar{X})$

$$\bar{r} \in \mathcal{Z} \Leftrightarrow \bar{r}(x, h) = R(x, h_1, h_2)\delta(h_3)$$
If $\bar{r} \in \mathcal{Z}$, then $(x, h, \xi, \nu) \in WF(\bar{r}) \Rightarrow h_3 = 0$. So source and receiver rays in $C_{\bar{F}}$ must terminate at same depth, to hit such a point.

Because of DSR assumption, this fixes the traveltimes $t_s, t_r$.

**Restricted to $\mathcal{Z}$, $C_{\bar{F}}$ is injective.**

$$\Rightarrow C_{\bar{F}*F} = I$$

$$\Rightarrow \bar{F}*\bar{F} is \Psi DO when restricted to $\mathcal{Z}$. $$
Quantifying the semblance principle: devise operator $W$ for which

$$\ker W \simeq \mathcal{R}_\chi,$$

then minimize a suitable norm of

$$W \bar{G}d^{\text{obs}}.$$

Converts inverse problem to optimization problem. With proper choice of $W$, $\Psi$DO property $\Rightarrow$ objective is smooth $\Rightarrow$ can use Newton and relatives.

**Upshot:** Claerbout’s depth oriented extension appears to offer basis for efficient new algorithms to solve velocity analysis problem - research currently under way in several groups.
Summary:

- quite a bit is known about the imaging problem under “standard hypotheses”: mathematics of multipathing imaging (asymptotic inversion, invertible extensions) clarified over last 10 years.
- many imaging situations (eg. near salt) violate “standard hypotheses” grossly - need much better theory
- extension of imaging via multiple suppression - some progress, many open questions re non-surface multiples
- velocity analysis - some progress, but still in primitive state mathematically
- almost no progress on underlying nonlinear inverse problem