Basic Periodic Homogenization

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Outline

Harmonic Average Rule for a 1D Problem

the two-point boundary value problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big(a^\epsilon(x)\frac{\mathrm{d}u^\epsilon}{\mathrm{d}x}\Big) = f \quad x \in \Omega = [0, L]$$
$$u^\epsilon(0) = u^\epsilon(L) = 0$$

assume $a^\epsilon(x) = a(x/\epsilon)$ and $a(y)$ is smooth and periodic with period 1

$$0 < \alpha \leq a(y) \leq \beta, \quad \forall y \in [0, 1]$$

we want to study the behavior of u^ϵ, as $\epsilon \to 0$

Harmonic Average Rule for a 1D Problem

from the variational formulation

$$\int_\Omega a^\epsilon \frac{\mathrm{d}u^\epsilon}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x, \qquad \forall v \in H_0^1(\Omega)$$

taking $v = u^\epsilon$

$$\Rightarrow \|u^\epsilon\|_{H^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}, \quad \forall \epsilon > 0$$

extracting a subsequence, still denoted by u^ϵ such that

$$u^\epsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega)$$

Harmonic Average Rule for a 1D Problem

introduce

$$\xi^\epsilon = a^\epsilon \frac{\mathrm{d}u^\epsilon}{\mathrm{d}x}$$

$\xi^\epsilon \in L^2(\Omega)$ since $a^\epsilon \in L^\infty(\Omega)$, then

$$-\frac{\mathrm{d}\xi^\epsilon}{\mathrm{d}x} = f$$

so $\{\xi^\epsilon\} \subseteq H^1(\Omega)$ and bounded. we can extract a subsequence, still denoted by ξ^ϵ such that

$$\xi^\epsilon \to \xi \quad \text{strongly in } L^2(\Omega)$$

by **Rellich compactness theorem**

Harmonic Average Rule for a 1D Problem

Def: if $g^\epsilon, g \in L^\infty(\Omega)$, $g^\epsilon \overset{*}{\rightharpoonup} g$ weak-* in $L^\infty(\Omega)$ means

$$\int_\Omega g^\epsilon \phi \, \mathrm{d}x = \int_\Omega g \phi \, \mathrm{d}x, \quad \forall \phi \in L^1(\Omega)$$

$\frac{1}{a^\epsilon} \overset{*}{\rightharpoonup} \mathcal{M}(a) := \int_0^1 \frac{1}{a(y)} \, \mathrm{d}y$ weak-* in $L^\infty(\Omega)$. then

$$\frac{1}{a^\epsilon}\xi^\epsilon \rightharpoonup \mathcal{M}(a)\xi \text{ weakly in } L^2(\Omega)$$

Harmonic Average Rule for a 1D Problem

$$\frac{\mathrm{d}u}{\mathrm{d}x} \leftharpoonup \frac{\mathrm{d}u^\epsilon}{\mathrm{d}x} = \frac{1}{a^\epsilon}\xi^\epsilon \rightharpoonup \mathcal{M}(a)\xi \text{ weakly in } L^2(\Omega) \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \mathcal{M}(a)\xi$$

since $-\frac{\mathrm{d}\xi}{\mathrm{d}x} = f$,

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{\mathcal{M}(a)}\frac{\mathrm{d}u}{\mathrm{d}x}\Big) = f$$

the homogenized operator is given by

$$\mathcal{A} = -\bar{a}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

where $\bar{a} = \dfrac{1}{\mathcal{M}(a)}$

Sobolev Space of Periodic Functions

- $\mathbb{T}^d$: d-dimensional unit cube, or called unit cell
- $f : \mathbb{R}^d \to \mathbb{R}$ called 1-**periodic** fun if

$$f(y + e_i) = f(y) \quad \forall y \in \mathbb{R}^d \quad i = 1, \cdots, \mathrm{d}$$

  where $\{e_i\}_{i=1}^d$ is the standard basis in $\mathbb{R}^d$
- $C^\infty_{per}(\mathbb{T}^d)$: the restriction to $\mathbb{T}^d$ of $C^\infty(\mathbb{R}^d)$ that are 1-periodic
- $L^p_{per}(\mathbb{T}^d)$: the completion of $C^\infty_{per}(\mathbb{T}^d)$ w.r.t. L^p-norm
- $H^1_{per}(\mathbb{T}^d) = \left\{u : u \in L^2_{per}(\mathbb{T}^d), \nabla u \in L^2_{per}(\mathbb{T}^d)\right\}$

Fredholm Alternative for Periodic Elliptic PDEs

consider the PDE

$$\mathcal{A}u = -\nabla \cdot \Big(A(y)\nabla u(y)\Big) = f(y), u(y) \text{ is } 1\text{−periodic}$$

Lemma. assume A is 1-periodic, uniformly coercive and bounded. then the following alternative holds.

i) either there exists a unique solution for every $f \in L^2_{per}(\mathbb{T}^d)$; or

ii) the homogeneous equation

$$\mathcal{A}u = 0, u \text{ is } 1\text{−periodic}$$

has at least one nontrivial solution and

$$1 \leq dim\Big(\mathcal{N}(\mathcal{A})\Big) = dim\Big(\mathcal{N}(\mathcal{A}^*)\Big) < \infty.$$

in this case the problem has a weak solution if and only if

$$(f, v)_{\mathbb{T}^d} = 0, \quad \forall v \in \mathcal{N}(\mathcal{A}^*)$$

More Applicable Corollary: Solvability Condition

Corollary. let $f(y) \in L^2_{per}(\mathbb{T}^d)$. there exists a solution in $H^1_{per}(\mathbb{T}^d)$ (unique up to an additive constant) of the elliptic PDE if and only if $\int_{\mathbb{T}^d} f(y)dy = 0$

Proof. indeed, consider the homogeneous adjoint equation

$$\mathcal{A}^* v = -\nabla \cdot \left(A^T \nabla v \right) = 0.$$

clearly, the constant function $v = 1$ is a solution of this equation. the uniform ellipticity of the matrix A implies that

$$\int_{\mathbb{T}^d} |\nabla v|^2 dy = 0$$

so that v is a constant. hence $\mathcal{N}(\mathcal{A}^*) = \mathrm{span}\{1\}$

Elliptic PDEs in d-Dimension

stationary diffusion equation in divergence form

$$-\nabla \cdot \left(A^\epsilon \nabla u^\epsilon \right) = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u^\epsilon = 0 \quad \text{on } \partial\Omega$$

- $u^\epsilon = u^\epsilon(x)$: an unknown scalar field
- $f = f(x)$: a given scalar field
- a coefficient tensor $A^\epsilon(x) = A(x/\epsilon) = A(y)$, A is 1-periodic w.r.t. y, uniformly coercive and bounded, i.e.,

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{d} A_{ij}(y)\xi_i\xi_j \leq \beta|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \forall y, \beta \geq \alpha > 0$$

Two-scale Asymptotic Expansions

- sol u^ϵ in the form of a power series expansion in ϵ

$$u^\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

- claim: $\{u_i\}$ depend explicitly on x and $y = x/\epsilon$ and 1-periodic w.r.t. y (idea of multiple scales)

$$\Rightarrow u^\epsilon(x) = u_0\Big(x, \frac{x}{\epsilon}\Big) + \epsilon u_1\Big(x, \frac{x}{\epsilon}\Big) + \epsilon^2 u_2\Big(x, \frac{x}{\epsilon}\Big) + \cdots$$

Two-scale Asymptotic Expansions

$y = x/\epsilon \Rightarrow \nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y$ e.g., $g^\epsilon(x) := g(x, \frac{x}{\epsilon})$

$$\nabla g^\epsilon(x) = \nabla_x g(x,y)\Big|_{y=\frac{x}{\epsilon}} + \frac{1}{\epsilon}\nabla_y g(x,y)\Big|_{y=\frac{x}{\epsilon}}$$

$\mathcal{A}^\epsilon := -\nabla \cdot (A(y)\nabla)$ in the form

$$\mathcal{A}^\epsilon = \frac{1}{\epsilon^2}\mathcal{A}_0 + \frac{1}{\epsilon}\mathcal{A}_1 + \mathcal{A}_2$$

where

$$\mathcal{A}_0 := -\nabla_y \cdot \Big(A(y)\nabla_y\Big)$$
$$\mathcal{A}_1 := -\nabla_y \cdot \Big(A(y)\nabla_x\Big) - \nabla_x \cdot \Big(A(y)\nabla_y\Big)$$
$$\mathcal{A}_2 := -\nabla_x \cdot \Big(A(y)\nabla_x\Big)$$

Two-scale Asymptotic Expansions

the equation becomes

$$\begin{aligned}\Big(\frac{1}{\epsilon^2}\mathcal{A}_0 + \frac{1}{\epsilon}\mathcal{A}_1 + \mathcal{A}_2\Big)u^\epsilon &= f \quad (x, y) \in \Omega \times \mathbb{T}^d \\ u^\epsilon &= 0 \quad (x, y) \in \partial\Omega \times \mathbb{T}^d\end{aligned}$$

moreover

$$\frac{1}{\epsilon^2}\mathcal{A}_0 u_0 + \frac{1}{\epsilon}(\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0) + (\mathcal{A}_0 u_2 + \mathcal{A}_1 u1 + \mathcal{A}_2 u_0) + \mathcal{O}(\epsilon) = f$$

Two-scale Asymptotic Expansions

disregard all terms of order higher than 1

$$
\begin{aligned}
\mathcal{O}(1/\epsilon^2) &\quad \mathcal{A}_0 u_0 = 0 \\
\mathcal{O}(1/\epsilon) &\quad \mathcal{A}_0 u_1 = -\mathcal{A}_1 u_0 \\
\mathcal{O}(1) &\quad \mathcal{A}_0 u_2 = -\mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 + f(x)
\end{aligned}
$$

$\mathcal{A}_0 u_0 = 0$

$u_0(x, y) = u(x)$, i.e., u_0 independent of $y \Leftarrow$ ellipticity of A

$$\alpha \int_{\mathbb{T}^d} |\nabla_y u_0(x,y)|^2 \,\mathrm{d}y \leq \int_{\mathbb{T}^d} A(y)\nabla_y u_0 \cdot \nabla_y u_0 \,\mathrm{d}y$$
$$= -\int_{\mathbb{T}^d} u_0 \mathcal{A}_0 u_0 \,\mathrm{d}y = 0$$

$\Longrightarrow \nabla_y u_0(x, y) = 0$

$\mathcal{A}_0 u_1 = -\mathcal{A}_1 u_0$

$\mathcal{A}_0 u_1 = \nabla_y \cdot (A \nabla_x u)$ and $u_1(x, y)$ is 1-periodic w.r.t. y, $\int_{\mathbb{T}^d} u_1 \, \mathrm{d}y = 0$ check the solvability condition

$$\int_{\mathbb{T}^d} \nabla_y \cdot (A \nabla_x u) \, \mathrm{d}y = \int_{\partial \mathbb{T}^d} \mathbf{n} \cdot (A \nabla_x u) \, \mathrm{d}S = 0$$

by periodicity of $A(\cdot)$

solving u_1

use separation of variables: $u_1(x,y) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(x)\omega_i(y)$ where $\chi(y) = \Big[\omega_1(y), \cdots, \omega_d(y)\Big]$ is called the **first-order corrector** and they satisfy the **cell problem**

$$
\begin{aligned}
& -\nabla_y \cdot \Big(A(y)\nabla_y \omega_i(y)\Big) = \nabla_y \cdot \Big(A(y)e_i\Big) \\
& \omega_i(y) \quad \text{is } 1-\text{periodic}
\end{aligned}
$$

where $\{e_i\}_{i=1}^{d}$ is the standard basis in $\mathbb{R}^d$

$\mathcal{A}_0 u_2 = -\mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 + f(x)$

the solvability condition of the $\mathcal{O}(1)$ equation implies

$$\int_{\mathbb{T}^d} (\mathcal{A}_2 u_0 + \mathcal{A}_1 u_1) \, \mathrm{d}y = f(x)$$

where

$$\begin{aligned} \int_{\mathbb{T}^d} \mathcal{A}_2 u_0 &= \int_{\mathbb{T}^d} -\nabla_x \cdot \Big(A(y) \nabla_x u \Big) \, \mathrm{d}y \\ &= -\nabla_x \cdot \Big[\Big(\int_{\mathbb{T}^d} A(y) \, \mathrm{d}y \Big) \nabla_x u \Big] \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{T}^d} \mathcal{A}_1 u_1 \, \mathrm{d}y &= \int_{\mathbb{T}^d} \Big(-\nabla_y \cdot (A(y) \nabla_x u_1) - \nabla_x \cdot (A(y) \nabla_y u_1) \Big) \, \mathrm{d}y \\ &:= I_1 + I_2 \end{aligned}$$

Continuing ...

- $I_1 = 0$ by periodicity and
- $$I_2 = \int_{\mathbb{T}^d} -\nabla_x \cdot (A(y)\nabla_y u_1)\,\mathrm{d}y$$
$$= -\int_{\mathbb{T}^d} \nabla_x \cdot \Big(A(y)\nabla_y(\chi \cdot \nabla_x u)\Big)\,\mathrm{d}y$$
$$= -\nabla_x \cdot \Big(\int_{\mathbb{T}^d} A(y)\nabla_y \chi(y)^T\,\mathrm{d}y\Big)\nabla_x u$$

finally, the homogenized equation

$$\nabla_x \cdot (\bar{A}\nabla_x u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

where

$$\bar{A} = \int_{\mathbb{T}^d} \Big(A(y) + A(y)\nabla\chi(y)^T\Big)\,\mathrm{d}y$$

Continuing ...

$$\nabla\chi(y)^T = \begin{pmatrix} \frac{\partial \omega_1}{\partial y_1} & \cdots & \frac{\partial \omega_1}{\partial y_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega_d}{\partial y_1} & \cdots & \frac{\partial \omega_d}{\partial y_d} \end{pmatrix}$$

and

$$(A(y)\nabla\chi(y)^T)_{ij} = \sum_k A_{ik}(y)\frac{\partial \omega_j}{\partial y_k}$$

Comments from Allaire's Slides

- explicit formula for the effective parameters but no longer true for non-periodic problems
- $\bar{A}$ not depend on ϵ, f, u or the boundary conditions, still true in the non-periodic case
- $\bar{A}$ is positive definite, but not necessary isotropic even if $A(y)$ was so
- one can check that

$$\lim_{\epsilon\to 0} u^\epsilon = u, \quad \lim_{\epsilon\to 0} \nabla u^\epsilon = \nabla u, \quad \lim_{\epsilon\to 0} A(\frac{x}{\epsilon})\nabla u^\epsilon = \bar{A}\nabla u$$

$$\lim_{\epsilon\to 0} A(\frac{x}{\epsilon})\nabla u^\epsilon \cdot \nabla u^\epsilon = \bar{A}\nabla u \cdot \nabla u$$

- same results for evolution problems
- very general method, but heuristic and not rigorous

1D Elliptic Problem

let $d = 1$ and $\Omega = [0, L]$. then the two-point boundary value problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big(a(\frac{x}{\epsilon})\frac{\mathrm{d}u^\epsilon}{\mathrm{d}x}\Big) = f \quad x \in [0, L]$$
$$u^\epsilon(0) = u^\epsilon(L) = 0$$

assume $a(y)$ is smooth and periodic with period 1 and

$$0 < \alpha \leq a(y) \leq \beta, \quad \forall y \in [0, 1]$$

1D Elliptic Problem

the cell problem in 1D

$$-\frac{\mathrm{d}}{\mathrm{d}y}\left(a(y)\frac{\mathrm{d}\chi}{\mathrm{d}y}\right) = \frac{\mathrm{d}a(y)}{\mathrm{d}y} \quad y \in [0,1]$$

$$\chi \text{ is } 1-\text{periodic}, \quad \int_{\mathbb{T}^d} \chi(y)\,\mathrm{d}y = 0(\Leftarrow \text{uniqueness})$$

integration from 0 to y

$$a(y)\frac{\mathrm{d}\chi(y)}{\mathrm{d}y} = -a(y) + c_1$$

once again

$$\chi(y) = -y + c_1\int_0^y \frac{1}{a(y)}\,\mathrm{d}y + c_2$$

$c_1 = \left(\int_0^1 1/a(y)\,\mathrm{d}y\right)^{-1}$ by periodicity of χ

1D Elliptic Problem

then the 1D effective coefficient - the *harmonic average*

$$\bar{a} = \int_0^1 \Big(a(y) + a(y) \frac{\mathrm{d}\chi(y)}{\mathrm{d}y} \Big) \,\mathrm{d}y = \Big(\int_0^1 1/a(y) \,\mathrm{d}y \Big)^{-1}$$

one can easily prove

$$\alpha \leq \bar{a} \leq \beta, \qquad \bar{a} \leq \int_0^1 a(y) \,\mathrm{d}y$$

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