Extensions and Nonlinear Inverse Scattering

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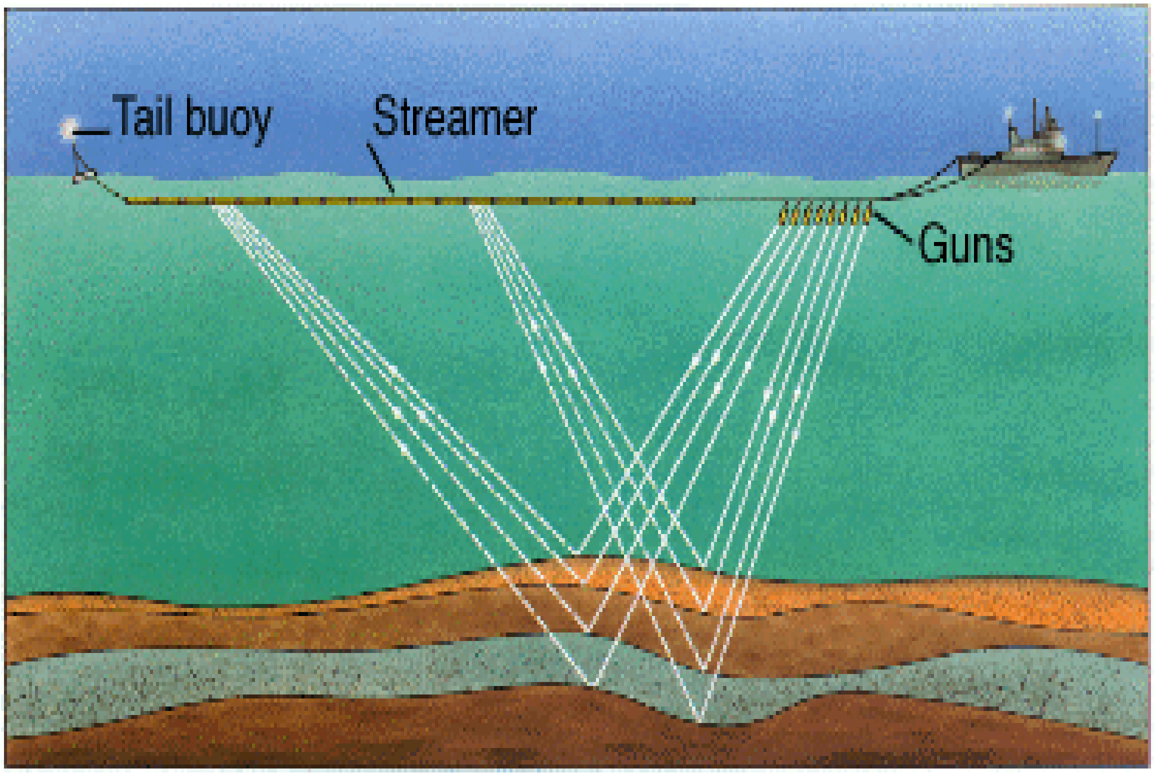
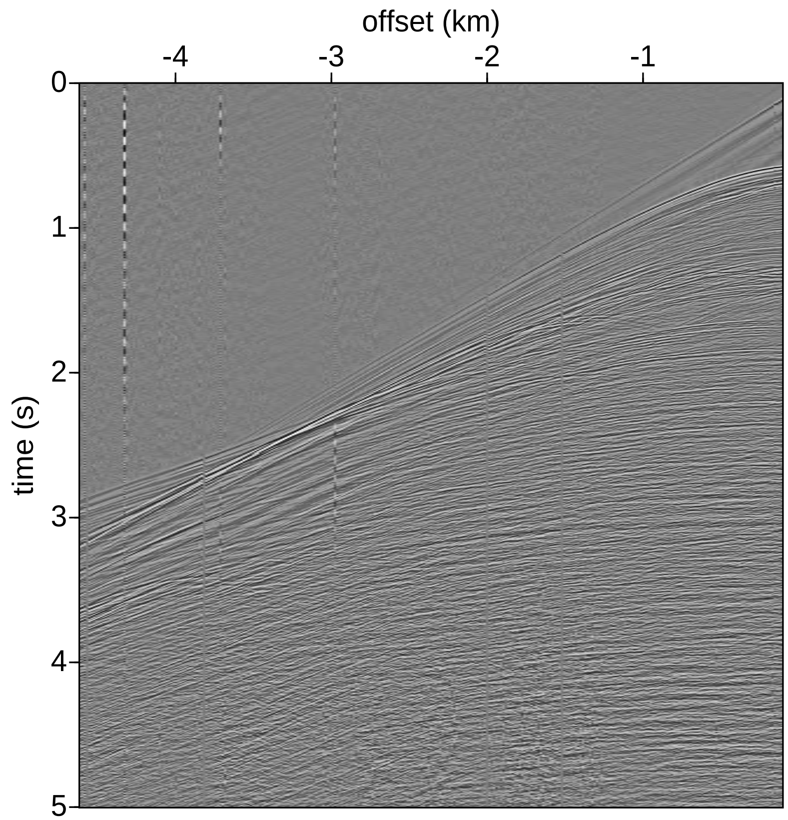


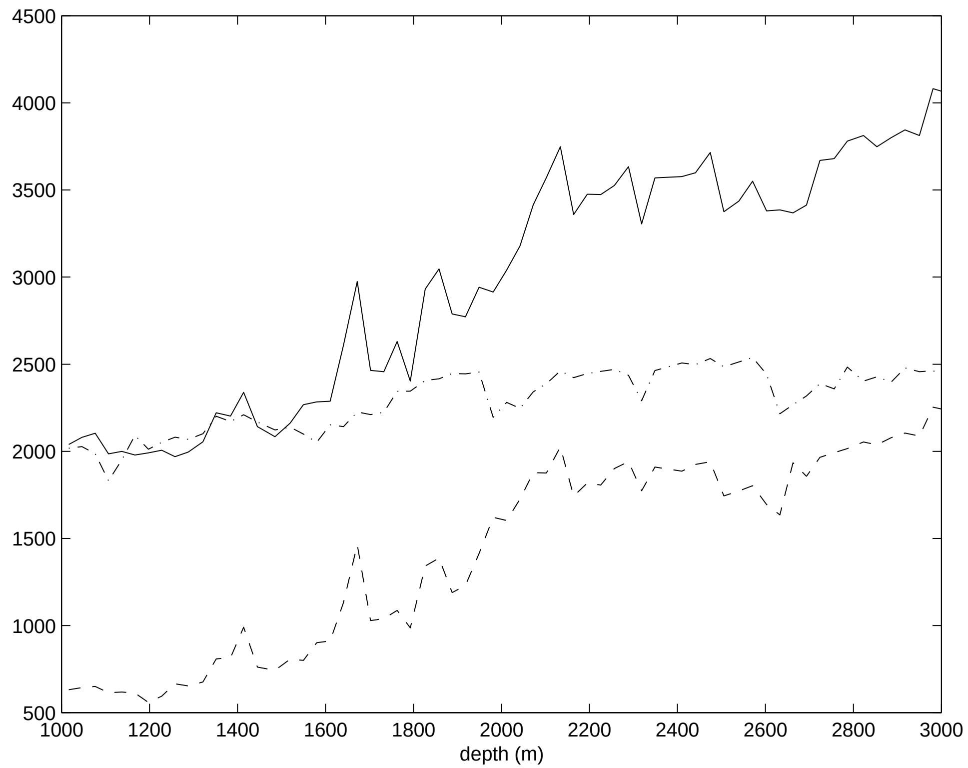
Data parameters: time t, source location $\mathbf{x}_s$, and receiver location $\mathbf{x}_r$, (vector) *half offset* $\mathbf{h} = \frac{\mathbf{x}_r - \mathbf{x}_s}{2}$, scalar half offset $h = |\mathbf{h}|$. Experiment = *shot*, single experiment data = *shot record*.

Typical Marine Record



Shot record, Gulf of Mexico (thanks: Exxon)

Mechanical Characteristics of Sedimentary Rock



Well logs from North Sea borehole. Top curve: v_p (m/s); middle curve: ρ (kg/m^3); bottom curve: v_s (m/s). (thanks: Mobil R&D, Viking Graben). Features: **large variance** on both short (wavelength) and long (km) distance scales.

Outline

1. A Model
2. Least Squares
3. Linearization
4. Extensions
5. Annihilators
6. Beyond Linearization

1. The Acoustic Model of Reflection Seismology

Constant Density Acoustic Model

acoustic potential $u(\mathbf{x},t)$, *sound velocity* $c(\mathbf{x})$ related to pressure p and particle velocity $\mathbf{v}$ by

$$p = \frac{\partial u}{\partial t}, \;\; \mathbf{v} = \frac{1}{\rho}\nabla u$$

Second order wave equation for potential

$$\left(\frac{1}{c(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\mathbf{x},t) = w(t)\delta(\mathbf{x}-\mathbf{x}_s)$$

plus initial, boundary conditions. RHS models localized energy source, "no low frequencies" - *many wavelengths* between source and target. *Useful idealization:* $w(t) = \delta(t)$, in which case $u = G(\mathbf{x}_s, \mathbf{x}, t)$(Green's function of the wave equation).

Forward map: $\mathcal{F}[c] \equiv p|_Y$, where $Y = \{(t, \mathbf{x}_r, \mathbf{x}_s) : 0 \leq t \leq T, ...\}$ is *acquisition manifold*.

2. Least Squares

Nonlinear inverse scattering

Inverse problem: given $d \in L^2(Y)$ find $c \in C$ s. t. $\mathcal{F}[c] \simeq d$.

A few questions:

- What is C?
- What is $\simeq$?
- If $\simeq$ means "close in L^2", could pose as *least squares* problem: find $c \in C$ as

$$c = \mathrm{argmin}\, \|\mathcal{F}[c] - d\|^2$$

Theory is inadequate - few rigorous answers to questions like these - but relevant properties of $\mathcal{F}$ understood *in broad outline*.

The bad news...

- Results of numerical experimentation disappointing (Tarantola 1986, many others)
- If δc is *smooth*, then $\mathcal{F}[c]$ and $\mathcal{F}[c+\delta c]$ tend to be *nearly orthogonal* even when δc is small $\Rightarrow$ least squares function tends to *saturate*, i.e. remain near its maximum, except when c is "right on average".
- fluctuations in angle between $\mathcal{F}[c]$, $\mathcal{F}[c+\delta c]$ as δc varies $\Rightarrow$ stationary points far from global min, *even when data is free of noise* $d = \mathcal{F}[c]$!!!
- Problems are so large that iterative methods (variants of Newton) are only feasible appraoch (3D: millions of unknowns, billions of equations) $\Rightarrow$ can only find stationary points;
- Therefore this approach *doesn't work*: it has had *no practical impact*.

3. Linearization

(Partly) linearized inverse scattering

Formally, $\mathcal{F}[v(1+r)] \simeq \mathcal{F}[v] + F[v]r$ where $F[\cdot]$ is *linearized forward map* defined by

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta G(\mathbf{x}_s,\mathbf{x},t) = 2\frac{r(\mathbf{x})}{v^2(\mathbf{x})}\frac{\partial^2 G}{\partial t^2}(\mathbf{x}_s,\mathbf{x},t)$$

$$F[v]r = \left.\frac{\partial \delta G}{\partial t}\right|_Y$$

- basis of most practical data processing procedures.
- v is no more known than r, inverse problem for $[v,r]$ still nonlinear!
- linearization error contains many effects observable in field data, notably **multiple reflections**, which can be quite strong, or even dominant - *so major open issue in this subject is how to go beyond linearization!!!*

Linearization error

Critical question: If there is any justice $F[v]r$ = directional derivative $D\mathcal{F}[v][vr]$ of $\mathcal{F}$ - but in what sense? Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

$$\mathcal{F}[v(1+r)] - (\mathcal{F}[v] + F[v]r)$$

- *small* when v smooth, r rough or oscillatory on wavelength scale - well-separated scales
- *large* when v not smooth and/or r not oscillatory - poorly separated scales

No mathematical results are known which justify/explain these observations in any rigorous way, except in 1D (Lewis & WWS IP 91).

The good news...

We actually know something about $F[v]$, besides its representation when $w(t) = \delta(t)$:

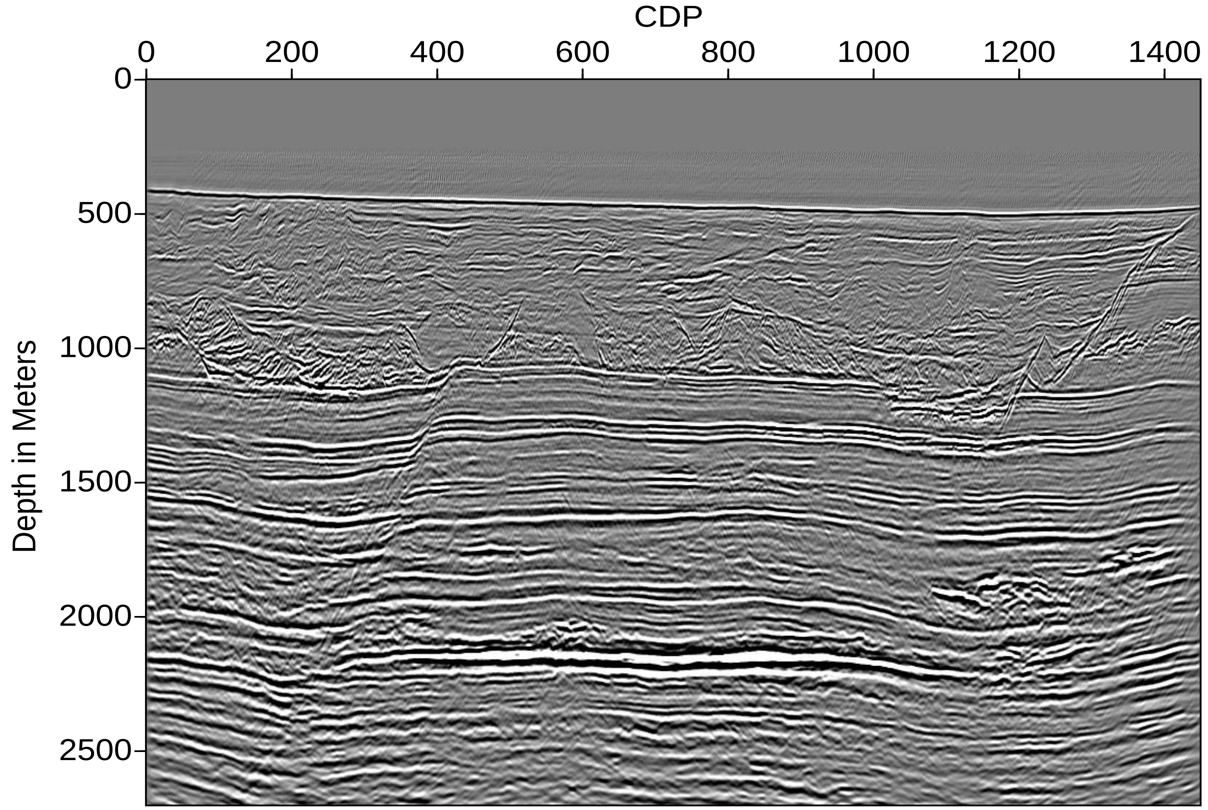
$$F[v]r(t, \mathbf{x}_r, \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau \, G(\mathbf{x}, \mathbf{x}_r, t-\tau) G(\mathbf{x}, \mathbf{x}_s, \tau) \frac{2r(\mathbf{x})}{v^2(\mathbf{x})}$$

Geometric optics provides asymptotic, high-frequency representations of G and these lead to oscillatory integral representation of $F[v]$. Consequences:

- rigorous results on solvability of least linear least squares problem ("linearized inversion") $\min_r \|F[v]r - (d - \mathcal{F}[v])\|^2$ (Beylkin 1985, Rakesh 1988, Smit et al. 1998, Nolan 1997, Stolk 2000),
- practical computational techniques - can represent $F[v]^\dagger$ as a *Generalized Radon Transform* (Beylkin 1985)

Knowledge of long model scales + data $\Rightarrow$ estimates of short model scales.

$$\min_r \|F[v]r - (d - \mathcal{F}[v])\|^2, \text{ given } v$$



Approximate linear least squares solution après Beylkin ("GRT inversion"), Mississippi Canyon, Gulf of Mexico, 2D survey (750 MB, 500 shots). Thanks: Exxon.

But what about v?

The long scale velocity model v is no more known that anything else, *a priori*.

Even if linearization assumed to be sufficiently accurate, the "partially linearized" least squares problem

$$\min_{v,r} \|F[v]r - (d - \mathcal{F}[v])\|^2$$

for v **and** r has same intractable character as fully nonlinear least squares inversion. Therefore this approach *doesn't work* either: it has had *no practical impact*.

[Aside: no, it doesn't help to measure error in some way other than L^2!]

So how are velocities found?

4. Extensions

Extended models

Extension of $F[v]$ (aka *extended model*): manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

$$
\begin{array}{ccccc}
 & & \bar{F}[v] & & \\
 & \mathcal{E}'(\bar{X}) & \rightarrow & \mathcal{D}'(Y) & \\
\chi & \uparrow & & \uparrow & \mathrm{id} \\
 & \mathcal{E}'(X) & \rightarrow & \mathcal{D}'(Y) & \\
 & & F[v] & &
\end{array}
$$

commutes, i.e.

$$\bar{F}[v]\chi r = F[v]r$$

Extension is "invertible" iff $\bar{F}[v]$ has a *right parametrix* $\bar{G}[v]$, i.e. $I - \bar{F}[v]\bar{G}[v]$is smoothing, or more generally if $\bar{F}[v]\bar{G}[v]$ is pseudodifferential ("inverse except for wrong amplitudes"). Also require existence of a left inverse η for χ: $\eta\chi = \mathrm{id}$.

NB: The trivial extension - $\bar{X} = X, \bar{F} = F$ - is virtually never invertible.

Grand Example

The Standard Extended Model: $\bar{X} = X \times H$, $H =$ offset range.

$\chi r(\mathbf{x},\mathbf{h}) = r(\mathbf{x})$, $\eta \bar{r}(\mathbf{x}) = \frac{1}{|H|}\int_H dh\, \bar{r}(\mathbf{x},\mathbf{h})$ ("stack").

$\bar{r} \in$ range of $\chi \Leftrightarrow$ plots of $\bar{r}(\cdot,\cdot,z,\mathbf{h})$ ("(prestack) image gathers") appear *flat*.

$$\bar{F}[v]\bar{r}(\mathbf{x}_r,\mathbf{x}_s,t) = \frac{\partial^2}{\partial t^2}\int dx \int d\tau\, G(\mathbf{x},\mathbf{x}_r,t-\tau)G(\mathbf{x},\mathbf{x}_s,\tau)\frac{2\bar{r}(\mathbf{x},\mathbf{h})}{v^2(\mathbf{x})}$$

(recall $\mathbf{h} = (\mathbf{x}_r - \mathbf{x}_s)/2$)

NB: $\bar{F}$ is "block diagonal" - family of operators (FIOs) parametrized by $\mathbf{h}$.

Reformulation of inverse problem

Given d, find v so that $\bar{G}[v]d \in$ the range of χ.

Claim: if v is so chosen, then $[v, r]$ solves partially linearized inverse problem with $r = \eta\bar{G}[v]d$.

Proof: Hypothesis means

$$\bar{G}[v]d = \chi r$$

for some r (whence necessarily $r = \eta\bar{G}[v]d$), so

$$d \simeq \bar{F}[v]\bar{G}[v]d = \bar{F}[v]\chi r = F[v]r$$

Q. E. D.

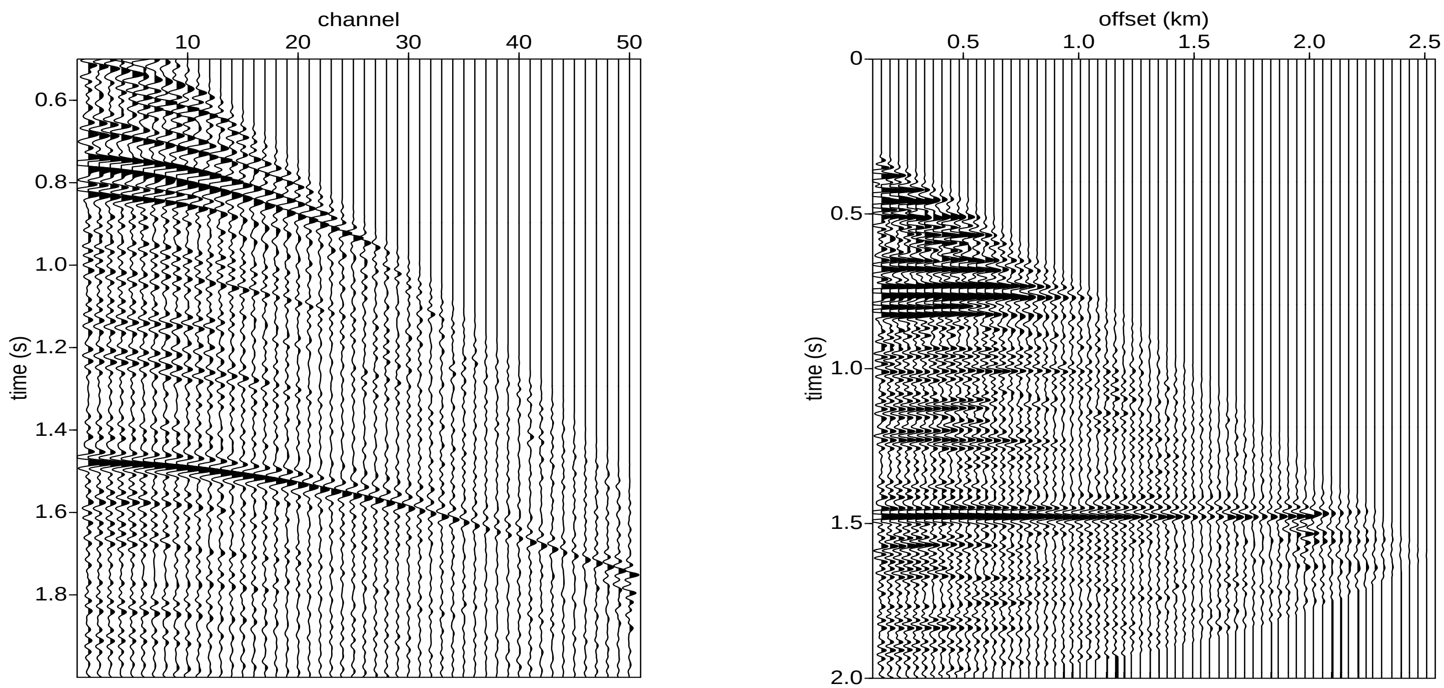
Application: Migration Velocity Analysis

Membership in range of χ is *visually evident*

$\Rightarrow$ industrial practice: adjust parameters of v *by hand* (!) until visual characteristics of $\mathcal{R}(\chi)$ satisfied - "flatten the image gathers".

For the Standard Extended Model, this means: until $\bar{G}[v]d$ is independent of $\mathbf{h}$.

Practically: insist only that $\bar{F}[v]\bar{G}[v]$ be pseudodifferential, so adjust v until $\bar{G}[v]d$ is "smooth" in $\mathbf{h}$.



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Left: shot record (d) from North Sea survey (thanks: Shell Research), lightly pre-processed.
Right: restriction of $\bar{G}[v]d^{\mathrm{obs}}$ to $x, y = \mathrm{const}$ (function of depth, offset): shows rel. sm'ness in h (offset) for properly chosen v.

5. Annihilators

Automating the reformulation

Suppose $W : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Z)$ annihilates range of χ:

$$\mathcal{E}'(X) \overset{\chi}{\to} \mathcal{E}'(\bar{X}) \overset{W}{\to} \mathcal{D}'(Z) \to 0$$

and moreover W is bounded on $L^2(\bar{X})$. Then

$$J[v;d] = \frac{1}{2}\|W\bar{G}[v]d\|^2$$

minimized when $[v, \eta\bar{G}[v]d]$ solves partially linearized inverse problem.

Construction of *annihilator* of $\mathcal{R}(F[v])$ (Guillemin, 1985):

$$d \in \mathcal{R}(F[v]) \Leftrightarrow \bar{G}[v]d \in \mathcal{R}(\chi) \Leftrightarrow W\bar{G}[v]d = 0$$

Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

- $$W = (I - \Delta)^{-\frac{1}{2}} \nabla_{\mathbf{h}}$$

  ("differential semblance" - WWS, 1986)

- $$W = I - \frac{1}{|H|} \int dh$$

  ("stack power" - Toldi, 1985)

- $$W = I - \chi F[v]^{\dagger} \bar{F}[v]$$

  $\Rightarrow$ minimizing $J[v, d]$ equivalent to reduced least squares.

But not many are good for much...

Since *problem is huge and data is noisy*, only W giving rise to differentiable $v, d \mapsto J[v, d]$ are useful - must be able to use Newton!!! Once again, idealize $w(t) = \delta(t)$.

Theorem (Stolk & WWS, 2003): $v, d \mapsto J[v, d]$ smooth $\Leftrightarrow$ W pseudodifferential.

i.e. only *differential semblance* gives rise to smooth optimization problem even with noisy data.

Some theory, many successful numerical tests of differential semblance using synthetic and field data: WWS et al., Chauris & Noble 2001, Mulder & tenKroode 2002. deHoop et al. 2004.

6. Beyond linearization

Invertible Extensions

Beylkin (1985), Rakesh (1988): if $\|\nabla^2 v\|_{C^0}$ "not too big" (no caustics appear), then the Standard Extension is invertible.

Nolan & WWS 1997, Stolk & WWS 2004: if $\|\nabla^2 v\|_{C^0}$ is too big (caustics, multi-pathing), Standard Extension is **not** invertible! Not in any version - common offset, common source, common scattering angle,...

Brings the whole program to a screeching halt, unless there are *other, inequivalent extensions*.

Claerbout's extension

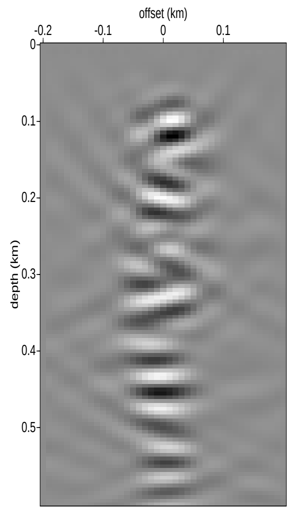
$\chi r(\mathbf{x},\mathbf{h}) = r(\mathbf{x})\delta(\mathbf{h})$, $\eta\bar{r}(\mathbf{x})$ "=" $\bar{r}(\mathbf{x},\mathbf{0})$ (Claerbout's zero-offset imaging condition)

$\bar{r} \in$ range of $\chi \Leftrightarrow$ plots of $\bar{r}(\cdot,\cdot,z,h)$ (i.e. *image gathers*) appear *focussed* at $\mathbf{h} = 0$

$$\bar{F}[v]\bar{r}(\mathbf{x}_r,\mathbf{x}_s,t) = \frac{\partial^2}{\partial t^2}\int dx \int dh \int d\tau\, G(\mathbf{x}+\mathbf{h},\mathbf{x}_r,t-\tau)G(\mathbf{x}-\mathbf{h},\mathbf{x}_s,\tau)\frac{2\bar{r}(\mathbf{x},\mathbf{h})}{v^2(\mathbf{x})}$$

This extension is invertible, assuming (i) $\bar{r}(\mathbf{x},\mathbf{h}) = \hat{r}(\mathbf{x},h_1,h_2)\delta(h_3)$ (horizontal offset only) and (ii) "DSR hypothesis": waves propagate up and down, not sideways ("rays do not turn") [Stolk-DeHoop 2001] and sometimes under more general conditions [WWS 2003].

Focussing at the right velocity



Offset Image Gather, x=1 km

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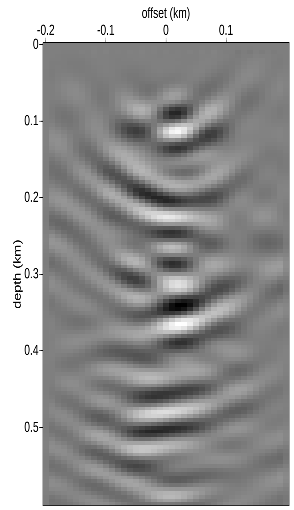

OIG, x=1 km: vel 10% high

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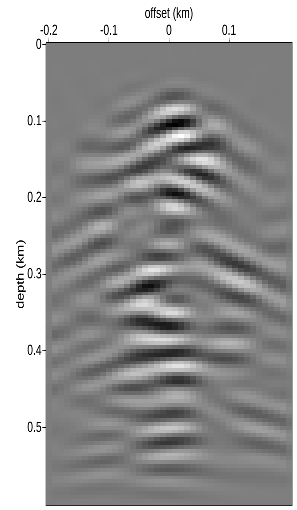

OIG, x=1 km: vel 10% low

Claerbout extension inverse ($\bar{G}$) applied to data from random r, constant v. From left to right: correct v, 10% high, 10% low. Observe **focussing** at $\mathbf{h} = 0$ for correct v.

Differential Semblance for Claerbout's Extension

$$W\bar{r}(\mathbf{x},\mathbf{h}) = \mathbf{h}\bar{r}(\mathbf{x},\mathbf{h}), \;\; J[v,d] = \frac{1}{2}\|W\bar{G}[v]d\|^2$$

Same smoothness properties as DS for Standard Extension.

P. Shen (2004): implementation, optimization via quasi-Newton algorithm, synthetic and field data.

Conclusion: successfully estimates v in settings (strong refraction) in which Standard Extension based DS fails.

Claerbout's Extension as a linearization

Write differential equation for $\bar{F}[v]$, by applying wave operator to both sides of integral representation: $\bar{F}[v]r = \delta\bar{u}|_Y$ where

$$\left(v^{-2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta\bar{u}(\mathbf{x},\mathbf{x}_s,t) = \int_H dh\, 2\bar{r}(\mathbf{x}-\mathbf{h},\mathbf{h})v^{-2}(\mathbf{x}-\mathbf{h})\frac{\partial^2 G}{\partial t^2}(\mathbf{x}-2\mathbf{h},\mathbf{x}_s,t)$$

Observe that this equation describes the linearization of the system

$$V^{-2}\left[\frac{\partial^2 u}{\partial t^2}\right] - \nabla^2 u(\mathbf{x},\mathbf{x}_s,t) = w(t)\delta(\mathbf{x}-\mathbf{x}_s),$$

in which the "velocity" V is an *operator*: formally,

$$Vw(\mathbf{x}) = \int_H dh\, K_V(\mathbf{x}-\mathbf{h},\mathbf{h})w(\mathbf{x}-2\mathbf{h})$$

and the linearization takes place at V with $K_V(\mathbf{x},\mathbf{h}) = v(\mathbf{x})\delta(\mathbf{h}) = \chi v(\mathbf{x},\mathbf{h})$.

The Nonlinear Claerbout Extension

That is, you can view Claerbout's extension of the linearized scattering problem as the linearization of an extension of the original scattering problem:

$$v^{-2}\left[\frac{\partial^2 u}{\partial t^2}\right] - \nabla^2 u(\mathbf{x},\mathbf{x}_s,t) = w(t)\delta(\mathbf{x}-\mathbf{x}_s),$$

where v is the operator of multiplication by the positive function v, *versus*

$$V^{-2}\left[\frac{\partial^2 u}{\partial t^2}\right] - \nabla^2 u(\mathbf{x},\mathbf{x}_s,t) = w(t)\delta(\mathbf{x}-\mathbf{x}_s),$$

with *self-adjoint positive* V.

This generalized nonlinear scattering problem makes sense: J.-L. Lions showed in the late '60s how to demonstrate the well-posedness of the initial value problem for operators like the above, with self-adjoint positive operator coefficients [also Stolk 2000].

Extended Inverse Scattering

The extended inverse scattering problem takes the place of the right inverse map $\bar{G}$ of the linear Claerbout extension: define the *extended forward map* $\bar{\mathcal{F}}$ by $\bar{\mathcal{F}}[V] = u|_Y$, where u solves

$$V^{-2}\left[\frac{\partial^2 u}{\partial t^2}\right] - \nabla^2 u(\mathbf{x},\mathbf{x}_s,t) = w(t)\delta(\mathbf{x}-\mathbf{x}_s),$$

plus appropriate initial and boundary conditions. Given a nominal noise level ϵ, an ϵ-solution of the extended inverse scattering problem is a positive self-adjoint V so that

$$\|\bar{\mathcal{F}}[V] - d\| \leq \epsilon \qquad (1)$$

In itself, this problem is grossly underdetermined - so use it as a constraint!

Nonlinear Differential Semblance

The *nonlinear differential semblance* problem is: given d, ϵ, find V to minimize

$$J[V, d, \epsilon] \equiv \|W K_V\|^2$$

subject to the constraint (1), where $W =$ multiply by $\mathbf{h}$ and K_V is the distribution kernel of V.

This problem statement combines the differential semblance automation of industrial velocity analysis with modeling of the nonlinear effects (multiple reflections etc.) observable in actual data.

Many open questions:

- What is a good class of operators? Must have well-behaved kernels!
- How to sensibly define the norm in J
- etc.

Conclusion

- Straightforward least squares formulation of (waveform) reflection seismic inverse problem *intractable* - very irregular with large residual stationary points $\Rightarrow$ *no influence on practice*.
- Linearized *extensions* provide framework for both (industry standard) interpretive velocity analysis and automated techniques based on construction of *range annihilators* - reformulation of inverse problem.
- Only *(pseudo)differential annihilators* yield smooth objective functions, successful automatic solution of partially linearized inverse problem.
- Claerbout's extension suitable for use in "complex structure" (strong refraction).
- Claerbout's extension also has a nonlinear generalization $Rightarrow$ approach the the full nonlinear inverse scattering problem.

Will it work? Stay tuned!

Thanks to...

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All of you, for listening

http://www.trip.caam.rice.edu