Extensions and Nonlinear Inverse Scattering

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Data parameters: time $t$, source location $x_s$, and receiver location $x_r$, (vector) half offset $h = \frac{x_r - x_s}{2}$, scalar half offset $h = |h|$. Experiment = shot, single experiment data = shot record.
Typical Marine Record

Shot record, Gulf of Mexico (thanks: Exxon)
Mechanical Characteristics of Sedimentary Rock

Well logs from North Sea borehole. Top curve: $v_p$ (m/s); middle curve: $\rho$ (kg/m$^3$); bottom curve: $v_s$ (m/s). (thanks: Mobil R&D, Viking Graben). Features: large variance on both short (wavelength) and long (km) distance scales.
Outline

1. A Model
2. Least Squares
3. Linearization
4. Extensions
5. Annihilators
6. Beyond Linearization
1. The Acoustic Model of Reflection Seismology
Constant Density Acoustic Model

*acoustic potential* \(u(x, t)\), *sound velocity* \(c(x)\) related to pressure \(p\) and particle velocity \(v\) by

\[
p = \frac{\partial u}{\partial t}, \quad v = \frac{1}{\rho} \nabla u
\]

Second order wave equation for potential

\[
\left( \frac{1}{c(x)^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) u(x, t) = w(t) \delta(x - x_s)
\]

plus initial, boundary conditions. RHS models localized energy source, “no low frequencies” - *many wavelengths* between source and target. *Useful idealization:* \(w(t) = \delta(t)\), in which case \(u = G(x_s, x, t)(\text{Green’s function of the wave equation}).

*Forward map:* \(\mathcal{F}[c] \equiv p|_Y\), where \(Y = \{(t, x_r, x_s) : 0 \leq t \leq T, \ldots\}\) is acquisition manifold.
2. Least Squares
Nonlinear inverse scattering

Inverse problem: given $d \in L^2(Y)$ find $c \in C$ s. t. $\mathcal{F}[c] \simeq d$.

A few questions:

- What is $C$?
- What is $\simeq$?
- If $\simeq$ means “close in $L^2$”, could pose as least squares problem: find $c \in C$ as
  
  $$c = \arg\min \| \mathcal{F}[c] - d \|^2$$

Theory is inadequate - few rigorous answers to questions like these - but relevant properties of $\mathcal{F}$ understood in broad outline.
The bad news...

- Results of numerical experimentation disappointing (Tarantola 1986, many others)
- If $\delta c$ is smooth, then $\mathcal{F}[c]$ and $\mathcal{F}[c + \delta c]$ tend to be nearly orthogonal even when $\delta c$ is small $\Rightarrow$ least squares function tends to saturate, i.e. remain near its maximum, except when $c$ is “right on average”.
- Fluctuations in angle between $\mathcal{F}[c]$, $\mathcal{F}[c + \delta c]$ as $\delta c$ varies $\Rightarrow$ stationary points far from global min, even when data is free of noise $d = \mathcal{F}[c]$!!
- Problems are so large that iterative methods (variants of Newton) are only feasible approach (3D: millions of unknowns, billions of equations) $\Rightarrow$ can only find stationary points;
- Therefore this approach doesn’t work: it has had no practical impact.
3. Linearization
(Partly) linearized inverse scattering

Formally, $\mathcal{F}[v(1+r)] \simeq \mathcal{F}[v] + F[v]r$ where $F[.]$ is linearized forward map defined by

$$
\left( \frac{1}{v(x)^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta G(x_s, x, t) = \frac{2r(x)}{v^2(x)} \frac{\partial^2 G}{\partial t^2}(x_s, x, t)
$$

$$
F[v]r = \left. \frac{\partial \delta G}{\partial t} \right|_Y
$$

- basis of most practical data processing procedures.
- $v$ is no more known than $r$, inverse problem for $[v, r]$ still nonlinear!
- linearization error contains many effects observable in field data, notably multiple reflections, which can be quite strong, or even dominant - so major open issue in this subject is how to go beyond linearization!!!
Linearization error

Critical question: If there is any justice $F[v]r = \text{directional derivative } D\mathcal{F}[v][vr]$ of $\mathcal{F}$ - but in what sense? Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

$$\mathcal{F}[v(1 + r)] - (\mathcal{F}[v] + F[v]r)$$

- *small* when $v$ smooth, $r$ rough or oscillatory on wavelength scale - well-separated scales
- *large* when $v$ not smooth and/or $r$ not oscillatory - poorly separated scales

No mathematical results are known which justify/explain these observations in any rigorous way, except in 1D (Lewis & WWS IP 91).
The good news...

We actually know something about $F[v]$, besides its representation when $w(t) = \delta(t)$:

$$F[v]r(t, x_r, x_s) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau \, G(x, x_r, t - \tau) G(x, x_s, \tau) \frac{2r(x)}{v^2(x)}$$

Geometric optics provides asymptotic, high-frequency representations of $G$ and these lead to oscillatory integral representation of $F[v]$. Consequences:

- rigorous results on solvability of least linear least squares problem (“linearized inversion”) $\min_r ||F[v]r - (d - \mathcal{F}[v])||^2$ (Beylkin 1985, Rakesh 1988, Smit et al. 1998, Nolan 1997, Stolk 2000),

- practical computational techniques - can represent $F[v]^\dagger$ as a Generalized Radon Transform (Beylkin 1985)

$\text{Knowledge of long model scales} + \text{data} \Rightarrow \text{estimates of short model scales.}$
\[ \min_r \| F[v]r - (d - \mathcal{F}[v]) \|^2, \] given \( v \)

Approximate linear least squares solution après Beylkin (“GRT inversion”), Mississippi Canyon, Gulf of Mexico, 2D survey (750 MB, 500 shots). Thanks: Exxon.
But what about $v$?

The long scale velocity model $v$ is no more known that anything else, \textit{a priori}.

Even if linearization assumed to be sufficiently accurate, the “partially linearized” least squares problem

$$
\min_{v,r} \| F[v]r - (d - \mathcal{F}[v]) \|^2
$$

for $v$ \textbf{and} $r$ has same intractable character as fully nonlinear least squares inversion. Therefore this approach \textit{doesn’t work} either: it has had \textit{no practical impact}.

[Aside: no, it doesn’t help to measure error in some way other than $L^2$!]

So how are velocities found?
4. Extensions
Extended models

*Extension of* $F[v]$ (aka *extended model*): manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

\[
\begin{array}{c}
\mathcal{E}'(\bar{X}) \\
\chi \uparrow \\
\mathcal{E}'(X)
\end{array} \quad \begin{array}{c}
\mathcal{D}'(Y) \\
\chi \uparrow \\
\mathcal{D}'(Y)
\end{array}
\]

\[\bar{F}[v] \text{ commutes, i.e.} \quad \bar{F}[v] \chi r = F[v]r\]

Extension is “invertible” iff $\bar{F}[v]$ has a *right parametrix* $\bar{G}[v]$, i.e. $I - \bar{F}[v]\bar{G}[v]$ is smoothing, or more generally if $\bar{F}[v]\bar{G}[v]$ is pseudodifferential (“inverse except for wrong amplitudes”). Also require existence of a left inverse $\eta$ for $\chi$: $\eta\chi = \text{id}$.

**NB:** The trivial extension - $\bar{X} = X$, $\bar{F} = F$ - is virtually never invertible.
Grand Example

The Standard Extended Model: $\tilde{X} = X \times H$, $H =$ offset range.

\[ \chi r(x, h) = r(x), \eta \bar{r}(x) = \frac{1}{|H|} \int_{H} dh \bar{r}(x, h) \text{ ("stack").} \]

$\bar{r} \in \text{range of } \chi \iff \text{plots of } \bar{r}(\cdot, \cdot, z, h) \text{ ("(prestack) image gathers") appear flat.}$

\[ \bar{F}[v] \bar{r}(x_r, x_s, t) = \frac{\partial^2}{\partial t^2} \int dx \int d\tau G(x, x_r, t - \tau)G(x, x_s, \tau) \frac{2\bar{r}(x, h)}{v^2(x)} \]

(recall $h = (x_r - x_s)/2$)

**NB:** $\bar{F}$ is "block diagonal" - family of operators (FIOs) parametrized by $h$. 
Reformulation of inverse problem

Given $d$, find $v$ so that $\bar{G}[v]d \in$ the range of $\chi$.

Claim: if $v$ is so chosen, then $[v, r]$ solves partially linearized inverse problem with $r = \eta\bar{G}[v]d$.

Proof: Hypothesis means

$$\bar{G}[v]d = \chi r$$

for some $r$ (whence necessarily $r = \eta\bar{G}[v]d$), so

$$d \simeq \bar{F}[v]\bar{G}[v]d = \bar{F}[v]\chi r = F[v]r$$

Q. E. D.
Application: Migration Velocity Analysis

Membership in range of $\chi$ is *visually evident*

$\Rightarrow$ industrial practice: adjust parameters of $v$ *by hand* (!) until visual characteristics of $R(\chi)$ satisfied - “flatten the image gathers”.

For the Standard Extended Model, this means: until $\bar{G}[v]d$ is independent of $h$.

Practically: insist only that $\bar{F}[v]\bar{G}[v]$ be pseudodifferential, so adjust $v$ until $\bar{G}[v]d$ is “smooth” in $h$. 
Left: shot record \((d)\) from North Sea survey (thanks: Shell Research), lightly pre-processed.

Right: restriction of \(\bar{G}[v]d^{obs}\) to \(x, y = \text{const}\) (function of depth, offset): shows rel. sm’ness in \(h\) (offset) for properly chosen \(v\).
5. Annihilators
Automating the reformulation

Suppose $W : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Z)$ annihilates range of $\chi$:

$$\begin{align*}
\chi & \quad \quad W \\
\mathcal{E}'(X) & \to \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Z) \to 0
\end{align*}$$

and moreover $W$ is bounded on $L^2(\bar{X})$. Then

$$J[v; d] = \frac{1}{2} \|W \bar{G}[v]d\|^2$$

minimized when $[v, \eta \bar{G}[v]d]$ solves partially linearized inverse problem.

Construction of annihilator of $\mathcal{R}(F[v])$ (Guillemin, 1985):

$$d \in \mathcal{R}(F[v]) \iff \bar{G}[v]d \in \mathcal{R}(\chi) \iff W \bar{G}[v]d = 0$$
Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

- \[ W = (I - \Delta)^{-\frac{1}{2}} \nabla h \]
  (“differential semblance” - WWS, 1986)

- \[ W = I - \frac{1}{|H|} \int dh \]
  (“stack power” - Toldi, 1985)

- \[ W = I - \chi F[v]^{\dagger} \bar{F}[v] \]
  \( \Rightarrow \) minimizing \( J[v, d] \) equivalent to reduced least squares.
But not many are good for much...

Since problem is huge and data is noisy, only $W$ giving rise to differentiable $v, d \mapsto J[v, d]$ are useful - must be able to use Newton!!! Once again, idealize $w(t) = \delta(t)$.

**Theorem** (Stolk & WWS, 2003): $v, d \mapsto J[v, d]$ smooth $\iff W$ pseudodifferential.

i.e. only *differential semblance* gives rise to smooth optimization problem even with noisy data.

6. Beyond linearization
Invertible Extensions

Beylkin (1985), Rakesh (1988): if $\| \nabla^2 v \|_C^0$ “not too big” (no caustics appear), then the Standard Extension is invertible.

Nolan & WWS 1997, Stolk & WWS 2004: if $\| \nabla^2 v \|_C^0$ is too big (caustics, multipathing), Standard Extension is not invertible! Not in any version - common offset, common source, common scattering angle,...

Brings the whole program to a screeching halt, unless there are other, inequivalent extensions.
Claerbout’s extension

\[ \chi r(x, h) = r(x)\delta(h), \eta \bar{r}(x) = \bar{r}(x, 0) \] (Claerbout’s zero-offset imaging condition)

\( \bar{r} \in \text{range of } \chi \iff \text{plots of } \bar{r}(\cdot, \cdot, z, h) \) (i.e. image gathers) appear focussed at \( h = 0 \)

\[
\bar{F}[\bar{v}] \bar{r}(x_r, x_s, t) = \frac{\partial^2}{\partial t^2} \int dx \int dh \int d\tau G(x+h, x_r, t-\tau)G(x-h, x_s, \tau) \frac{2\bar{r}(x, h)}{v^2(x)}
\]

This extension is invertible, assuming (i) \( \bar{r}(x, h) = \hat{r}(x, h_1, h_2)\delta(h_3) \) (horizontal offset only) and (ii) ”DSR hypothesis”: waves propagate up and down, not sideways (“rays do not turn”) [Stolk-DeHoop 2001] and sometimes under more general conditions [WWS 2003].
Claerbout extension inverse ($\widetilde{G}$) applied to data from random $r$, constant $v$. From left to right: correct $v$, 10% high, 10% low. Observe **focussing** at $h = 0$ for correct $v$. 
Differential Semblance for Claerbout’s Extension

\[ W\bar{r}(x, h) = h\bar{r}(x, h), \quad J[v, d] = \frac{1}{2}\|W\bar{G}[v]d\|^2 \]

Same smoothness properties as DS for Standard Extension.

P. Shen (2004): implementation, optimization via quasi-Newton algorithm, synthetic and field data.

Conclusion: successfully estimates \( v \) in settings (strong refraction) in which Standard Extension based DS fails.
Claerbout’s Extension as a linearization

Write differential equation for $\bar{F}[v]$, by applying wave operator to both sides of integral representation: $\bar{F}[v]r = \delta \bar{u}|_Y$ where

$\left(v^{-2} \frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta \bar{u}(x, x_s, t) = \int_H dh \, 2\bar{r}(x - h, h)v^{-2}(x - h) \frac{\partial^2 G}{\partial t^2}(x - 2h, x_s, t)$

Observe that this equation describes the linearization of the system

$V^{-2} \left[ \frac{\partial^2 u}{\partial t^2} \right] - \nabla^2 u(x, x_s, t) = w(t)\delta(x - x_s)$,

in which the “velocity” $V$ is an operator: formally,

$Vw(x) = \int_H \, dh \, K_V(x - h, h)w(x - 2h)$

and the linearization takes place at $V$ with $K_V(x, h) = v(x)\delta(h) = \chi v(x, h)$. 
The Nonlinear Claerbout Extension

That is, you can view Claerbout’s extension of the linearized scattering problem as the linearization of an extension of the original scattering problem:

\[ v^{-2} \left[ \frac{\partial^2 u}{\partial t^2} \right] - \nabla^2 u(x, x_s, t) = w(t)\delta(x - x_s), \]

where \( v \) is the operator of multiplication by the positive function \( v \), versus

\[ V^{-2} \left[ \frac{\partial^2 u}{\partial t^2} \right] - \nabla^2 u(x, x_s, t) = w(t)\delta(x - x_s), \]

with self-adjoint positive \( V \).

This generalized nonlinear scattering problem makes sense: J.-L. Lions showed in the late ’60s how to demonstrate the well-posedness of the initial value problem for operators like the above, with self-adjoint positive operator coefficients [also Stolk 2000].
Extended Inverse Scattering

The extended inverse scattering problem takes the place of the right inverse map $\tilde{G}$ of the linear Claerbout extension: define the extended forward map $\tilde{F}$ by $\tilde{F}[V] = u|_Y$, where $u$ solves

$$V^{-2} \left[ \frac{\partial^2 u}{\partial t^2} \right] - \nabla^2 u(x, x_s, t) = w(t)\delta(x - x_s),$$

plus appropriate initial and boundary conditions. Given a nominal noise level $\epsilon$, an $\epsilon$-solution of the extended inverse scattering problem is a positive self-adjoint $V$ so that

$$\|\tilde{F}[V] - d\| \leq \epsilon$$

(1)

In itself, this problem is grossly underdetermined - so use it as a constraint!
Nonlinear Differential Semblance

The nonlinear differential semblance problem is: given $d, \epsilon$, find $V$ to minimize

$$J[V, d, \epsilon] \equiv \|WK_V\|^2$$

subject to the constraint (1), where $W = \text{multiply by } h$ and $K_V$ is the distribution kernel of $V$.

This problem statement combines the differential semblance automation of industrial velocity analysis with modeling of the nonlinear effects (multiple reflections etc.) observable in actual data.

Many open questions:

- What is a good class of operators? Must have well-behaved kernels!
- How to sensibly define the norm in $J$
- etc.
Conclusion

• Straightforward least squares formulation of (waveform) reflection seismic inverse problem *intractable* - very irregular with large residual stationary points $\Rightarrow$ no influence on practice.

• Linearized *extensions* provide framework for both (industry standard) interpretive velocity analysis and automated techniques based on construction of *range annihilators* - reformulation of inverse problem.

• Only *(pseudo)differential annihilators* yield smooth objective functions, successful automatic solution of partially linearized inverse problem.

• Claerbout’s extension suitable for use in “complex structure” (strong refraction).

• Claerbout’s extension also has a nonlinear generalization $\Rightarrow$ approach the full nonlinear inverse scattering problem.

Will it work? Stay tuned!
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