Image amplitudes in reverse time migration/inversion

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TRIP Seminar, October 6, 2010

RTM based linearized inversion

- Seismic imaging is mathematically treated as a linearized inverse problem
- Kirchhoff migration can correspond to a method for inverting discontinuities (Beylkin, 1985).
- There are conditions on the background medium. Single source: no multipathing. Multisource: Traveltime Injectivity Condition (Rakesh, 1988; Nolan and Symes 1997; Ten Kroode Et Al., 1998)
- RTM has the advantage that there are no approximations from ray-theory and one-way methods. We will do linearized inverse scattering by a modified reverse time migration (RTM algorithm

The linearized inverse problem

Source field u_{src}

$$\left(\frac{1}{v^2}\partial_t^2 - \nabla_{\mathbf{x}}^2\right) u_{\mathrm{src}}(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_{\mathrm{src}})\delta(t)$$

Scattered field: Linearize in the velocity $\frac{1}{v(\mathbf{x})^2} \to \frac{1}{v(\mathbf{x})^2}(1 + r(\mathbf{x}))$.
v the smooth velocity model, r contains singularities

$$\left(\frac{1}{v^2}\partial_t^2 - \nabla_{\mathbf{x}}^2\right) u_{\mathrm{scat}}(\mathbf{x}, t) = -\frac{r}{v^2}\partial_t^2 u_{\mathrm{src}}.$$

Problem: Determine $r(\mathbf{x})$ from the following data:

$u_{\mathrm{scat}}(\mathbf{x}, t)$ for $\mathbf{x} = (x_1, x_2, x_3)$ in a subset of $x_3 = 0$, $t \in [0, T_{\max}]$

A new view on Reverse Time Migration

- Numerically solve wave equation to obtain
  $u_{\rm src}(\mathbf{x}, t)$ = incoming (source) wave field
  $u_{\rm rtc}(\mathbf{x}, t)$ = reverse time continued receiver field
- New imaging condition

$$I(\mathbf{x}) = \frac{2i}{2\pi\omega^3} \int \frac{\omega^2\, \overline{\widehat{u}_{\rm src}(\mathbf{x},\omega)}\widehat{u}_{\rm rtc}(\mathbf{x},\omega) - v^2\, \overline{\nabla\widehat{u}_{\rm src}(\mathbf{x},\omega)}\nabla\widehat{u}_{\rm rtc}(\mathbf{x},\omega)}{|\widehat{u}_{\rm src}(\mathbf{x},\omega)|^2}\, d\omega.$$

  (cf. Kiyashchenko et al., 2007)
- Characterization of I

$$I(\mathbf{x}) = R(\mathbf{x}, D_{\mathbf{x}})r,$$

  Here R is a pseudodifferential operator that describes the aperture effect, i.e. $R(\mathbf{x}, \boldsymbol{\xi}) = 1$ for "visible" reflectors (cf. Beylkin, 1985).

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Model problem

Notation $\mathbf{x} = (x_1, x_2, x_3)$.

Source field is plane wave with velocity $(0, 0, v)$

$$u_{\mathrm{src}}(\mathbf{x}, t) = A\,\delta(t - \frac{x_3}{v})$$

Scattered field slightly simplified

$$(\frac{1}{v^2}\partial_t^2 - \nabla_{\mathbf{x}}^2)u_{\mathrm{scat}}(t, \mathbf{x}) = A\,\delta(t - \frac{x_3}{v})r(\mathbf{x}).$$

Perfect backpropagation: u_{rtc} solution of a final value problem

$$(\frac{1}{v^2}\partial_t^2 - \nabla_{\mathbf{x}}^2)u_{\mathrm{rtc}}(\mathbf{x}, t) = 0$$

$$u_{\mathrm{rtc}}(\mathbf{x}, T_1) = u_{\mathrm{scat}}(\mathbf{x}, T_1), \qquad \partial_t u_{\mathrm{rtc}}(\mathbf{x}, T_1) = \partial_t u_{\mathrm{scat}}(\mathbf{x}, T_1), \qquad x \in \mathbb{R}^3.$$

Solving the wave equation

Spatial Fourier transform

$$(v^{-2}\partial_t^2 + \boldsymbol{\xi}^2)\,\widehat{u}(\boldsymbol{\xi}, t) = \widehat{f}(\boldsymbol{\xi}, t).$$

Duhamel's principle: Let $\widehat{w}(\boldsymbol{\xi}, t, s)$

$$(v^{-2}\partial_t^2 + \boldsymbol{\xi}^2)\widehat{w}(\boldsymbol{\xi}, t, s,) = 0,$$
$$\widehat{w}(\boldsymbol{\xi}, s, s) = 0 \qquad \partial_t \widehat{w}(\boldsymbol{\xi}, s, s) = v^2 \widehat{f}(s, \xi).$$

Result

$$\widehat{u}(\boldsymbol{\xi}, t) = \int_0^t \left(e^{iv\|\boldsymbol{\xi}\|(t-s)} - e^{-iv\|\boldsymbol{\xi}\|(t-s)} \right) \frac{v^2\,\widehat{f}(\boldsymbol{\xi}, s)}{2iv\|\boldsymbol{\xi}\|}\, ds$$

Needed: Fourier transform of r.h.s. $A\,\delta(t - \frac{x_3}{v}) r(\mathbf{x})$

Fourier transform $f(\mathbf{x}, t) = A\,\delta(t - \frac{x_3}{v}) r(\mathbf{x})$

$$\widehat{f}(\boldsymbol{\xi}, t) = \int A\,\delta(t - \frac{x_3}{v}) r(\mathbf{x}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x}$$
$$= vA\,\mathcal{F}_{(x_1,x_2)\mapsto(\xi_1,\xi_2)} r(\xi_1, \xi_2, v_0 t) e^{-i\xi_3 vt}.$$

Computations involve two similar terms, take only one of those.

$$\widehat{u}_{\mathrm{scat}}(\boldsymbol{\xi}, t) = \int_0^t \frac{v^3}{-2iv\|\boldsymbol{\xi}\|} e^{-iv\|\boldsymbol{\xi}\|(t-s)-i\xi_3 vs} A\,\mathcal{F}_{(x_1,x_2)\mapsto(\xi_1,\xi_2)} r(\xi_1, \xi_2, vs)\, ds$$
$$+ \text{pos. freq. term}$$
$$= \frac{Av^2}{-2iv\|\boldsymbol{\xi}\|} e^{-iv\|\boldsymbol{\xi}\|t} \widehat{r}(\xi_1, \xi_2, \xi_3 - \|\boldsymbol{\xi}\|) + \text{pos. freq. term},$$

if t is after the incoming has passed $\operatorname{supp}(r)$
Back to space domain

$$u_{\mathrm{scat}}(\mathbf{x}, t) = 2\,\mathrm{Re}\, \frac{1}{(2\pi)^3} \int \frac{Av^2}{-2iv\|\xi\|}\, e^{i\mathbf{x}\cdot\boldsymbol{\xi}-iv\|\boldsymbol{\xi}\|t} \widehat{r}(\boldsymbol{\xi}-(0, 0, \|\boldsymbol{\xi}\|))\, d\boldsymbol{\xi}$$

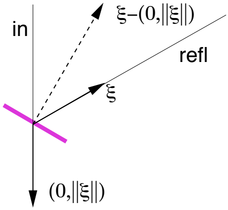
Interpretation

Scattered field

$$u_{\mathrm{scat}}(\mathbf{x}, t) = 2\,\mathrm{Re}\,\frac{1}{(2\pi)^3}\int \frac{Av^2}{-2iv\|\xi\|}\, e^{i\mathbf{x}\cdot\boldsymbol{\xi} - iv\|\boldsymbol{\xi}\|t}\,\widehat{r}(\boldsymbol{\xi} - (0,0,\|\boldsymbol{\xi}\|))\,d\boldsymbol{\xi}$$

Wave vectors

$\boldsymbol{\xi}$	outgoing wave number
$(0, 0, \|\boldsymbol{\xi}\|)$	incoming wave number
$\boldsymbol{\xi} - (0, 0, \|\boldsymbol{\xi}\|)$	reflectivity wave number



Modified excitation time imaging condition

Same formula for the backpropagated field! Valid for all t

$$u_{\rm rtc}(\mathbf{x}, t) = 2\,\mathrm{Re}\,\frac{1}{(2\pi)^3}\int \frac{Av^2}{-2iv\|\boldsymbol{\xi}\|}\, e^{i\mathbf{x}\cdot\boldsymbol{\xi} - iv\|\boldsymbol{\xi}\|t}\,\widehat{r}(\boldsymbol{\xi} - (0,0,\|\boldsymbol{\xi}\|))\,d\boldsymbol{\xi}$$

Basic image: $I_0(\mathbf{x}) = u_{\rm rtc}(\mathbf{x}, x_3/v)$.

Linearized inverse scattering: Try

$$I(\mathbf{x}) = \frac{2}{v^2 A}\left(\partial_t + v\partial_{x_3}\right) u_{\rm rtc}(\mathbf{x}, x_3/v)$$

(meaning first take derivatives then insert $(\mathbf{x}, t) = (\mathbf{x}, x_3/v)$)
Straightforward calculation yields

$$I(\mathbf{x}) = 2\,\mathrm{Re}\,\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(1 - \frac{\xi_3}{\|\boldsymbol{\xi}\|}\right)\widehat{r}(\boldsymbol{\xi} - (0,0,\|\boldsymbol{\xi}\|))\,d\boldsymbol{\xi}$$

A change of variables

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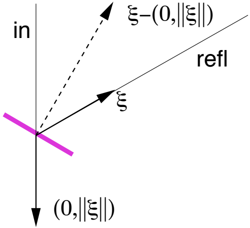
$$I(\mathbf{x}) = 2\,\mathrm{Re}\,\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(1-\frac{\xi_3}{\|\boldsymbol{\xi}\|}\right)\widehat{r}(\boldsymbol{\xi}-(0,0,\|\boldsymbol{\xi}\|))\,d\boldsymbol{\xi}$$

Change of variables

$$\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} - (0,0,\|\boldsymbol{\xi}\|)$$

- Jacobian $1-\frac{\xi_3}{\|\boldsymbol{\xi}\|}$
- Domain $\tilde{\boldsymbol{\xi}} \in \mathbb{R}^2 \times \mathbb{R}_{<0}$



Result:

$$I(\mathbf{x}) = 2\,\mathrm{Re}\,\frac{1}{(2\pi)^3}\int_{\mathbb{R}^2\times\mathbb{R}_{<0}}\widehat{r}(\tilde{\boldsymbol{\xi}})e^{i\mathbf{x}\cdot\tilde{\boldsymbol{\xi}}}\,d\tilde{\boldsymbol{\xi}} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^2\times\mathbb{R}_{\neq 0}}\widehat{r}(\tilde{\boldsymbol{\xi}})e^{i\mathbf{x}\cdot\tilde{\boldsymbol{\xi}}}\,d\tilde{\boldsymbol{\xi}}.$$

Reconstruction except for $\xi_3 = 0$, corresponding to reflection over 180 degrees.

Modified ratio imaging condition

We had

$$I(\mathbf{x}) = \frac{2}{v^2 A}\left(\partial_t + (0,0,v)\cdot\nabla_{\mathbf{x}}\right) u_{\mathrm{rtc}}(\mathbf{x}, x_3/v)$$

Rewrite into modified ratio imaging condition

- Use that $(0,0,v) = v^2 \nabla T(\mathbf{x})$, and that

$$\widehat{u}_{\mathrm{src}}(\mathbf{x},\omega) = Ae^{-i\omega T(\mathbf{x})},$$
$$\nabla_{\mathbf{x}}\widehat{u}_{\mathrm{src}}(\mathbf{x},\omega) \approx -i\omega\nabla_{\mathbf{x}}T(\mathbf{x})Ae^{-i\omega T(\mathbf{x})}.$$

- Insert factors $\frac{v^2}{\omega^2}$ left out in model problem

This results in

$$I(\mathbf{x}) = \frac{2i}{2\pi\omega^3}\int \frac{\omega^2\,\overline{\widehat{u}_{\mathrm{src}}(\mathbf{x},\omega)}\widehat{u}_{\mathrm{rtc}}(\mathbf{x},\omega) - v^2\,\overline{\nabla\widehat{u}_{\mathrm{src}}(\mathbf{x},\omega)}\nabla\widehat{u}_{\mathrm{rtc}}(\mathbf{x},\omega)}{|\widehat{u}_{\mathrm{src}}(\mathbf{x},\omega)|^2}\,d\omega.$$

Extension to variable background

Local vs. global analysis

- A local analysis generalizes the explicit formulas to variable coefficients, and leads to explicit inversion formulas
- A global analysis takes into account the of global effect of curved rays. No source-multipathing is allowed to exclude kinematic artifacts ("cross talk"). Cf. Rakesh (1988), Nolan and Symes (1997).

Tools: microlocal analysis

- Fourier integral operators (FIO's), pseudodifferential operators (ΨDO's)
- Description of the mapping properties of operators for localized plane wave components. Work in $(\mathbf{x}, \boldsymbol{\xi})$ or $(\mathbf{x}, t, \boldsymbol{\xi}, \omega)$ domain.

Variable coefficients: The wave equation

Local analysis

Solve the wave equation using WKB with plane wave initial values. Result is an FIO

$$u(\mathbf{y},t) = \frac{1}{(2\pi)^3}\iint e^{i\alpha(\mathbf{y},t-s,\boldsymbol{\xi})} a(\mathbf{y},t-s,\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi},s)\,ds\,d\boldsymbol{\xi} + \text{pos. freq. term},$$

α satisfies the eikonal equation; a a transport equation. Formula is valid for t, s in a bounded time interval

Global analysis

- Propagation of localized plane wave components along rays, properly described as curves in $(\mathbf{x}, t, \boldsymbol{\xi}, \omega)$ space
- Time reversal property is globally valid

Continued scattered field and linearized forward map

Define the continued scattered field u_{h}, as the "perfect" backpropagated field from a fixed time, full position space

Local result

For a localized contribution to r

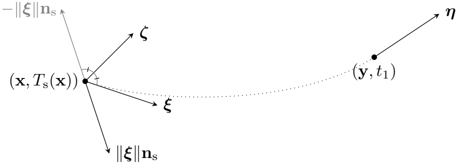
$$u_{\mathrm{h,a}}(\mathbf{y}, t) = \frac{1}{(2\pi)^3} \iint e^{i\varphi_T(\mathbf{y},t,\mathbf{x},\xi)} A(\mathbf{y}, t, \mathbf{x}, \boldsymbol{\xi})\, r(\mathbf{x})\, d\boldsymbol{\xi}\, d\mathbf{x}.$$

With phase function

$$\varphi_T(\mathbf{y}, t, \mathbf{x}, \xi) = \alpha(\mathbf{y}, t - T(\mathbf{x}), \boldsymbol{\xi}) - \boldsymbol{\xi} \cdot \mathbf{x}$$

Global result

The map $F_- : r \mapsto u_{\mathrm{h}}$ is a FIO. Mapping of wave components $(\mathbf{x}, \zeta) \mapsto (\mathbf{y}, t, \boldsymbol{\eta}, \omega)$: see picture



RT continuation from the boundary

Preprocess data:

- ΨDO cutoff $\Psi_M(y_1, y_2, t, \eta_1, \eta_2, \omega)$ with three effects
  - Smooth taper near acquisition boundary
  - Remove direct waves
  - Remove tangently incoming waves
- Normalize wave field to get true amplitude time reversal

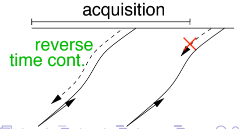
Characterize mathematically the aperture effect

Result There are ΨDO's P_-, P_+, such that

$$u_{\mathrm{rtc}}(\mathbf{x}, t) = P_-(t, x, D_x) u_{\mathrm{h},-} + P_+(t, x, D_x) u_{\mathrm{h},+}$$

in which $u_{\mathrm{h},\pm}$ is the part of u_{h} with $\pm\omega > 0$. To highest order

$P_\pm(s, \mathbf{x}, \boldsymbol{\xi}) = \Psi_M(y_1, y_2, t, \eta_1, \eta_2, \omega)$ if there is a ray with $\pm\omega > 0$, connecting $(\mathbf{x}, s, \boldsymbol{\xi}, \omega)$ with $(y_1, y_2, t, \eta_1, \eta_2, \omega)$



Inverse scattering

Modified excitation time imaging condition

$$I(\mathbf{x}) = \frac{2}{A_{\rm src}(\mathbf{x})}\partial_t^{-\frac{n+1}{2}}\left[\partial_t + v^2\nabla T\cdot\nabla_{\mathbf{x}}\right]u_{\rm rtc}(\mathbf{x},t)$$

- From the global analysis, there are no nonlocal contributions and

$$I(\mathbf{x}) = \Psi\text{DO}\ r(\mathbf{x})$$

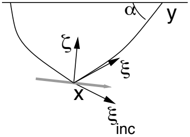
- From the local analysis

$$I(\mathbf{x}) = \left(R_-(\mathbf{x},D_{\mathbf{x}}) + R_+(\mathbf{x},D_{\mathbf{x}})\right) r(\mathbf{x})$$

To highest order

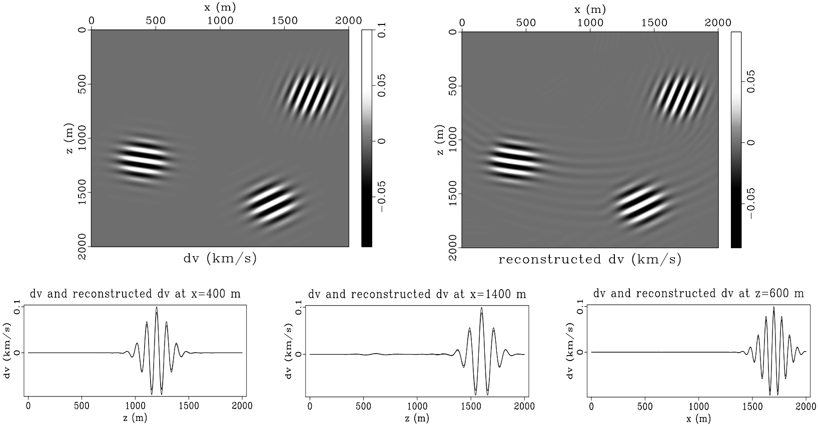
$$R_\pm(\mathbf{x},\zeta) = P_\pm(\mathbf{x},\boldsymbol{\xi})$$

if ζ and $\boldsymbol{\xi}$ are related through $\zeta = \boldsymbol{\xi} \pm \omega\nabla T(\mathbf{x})$.



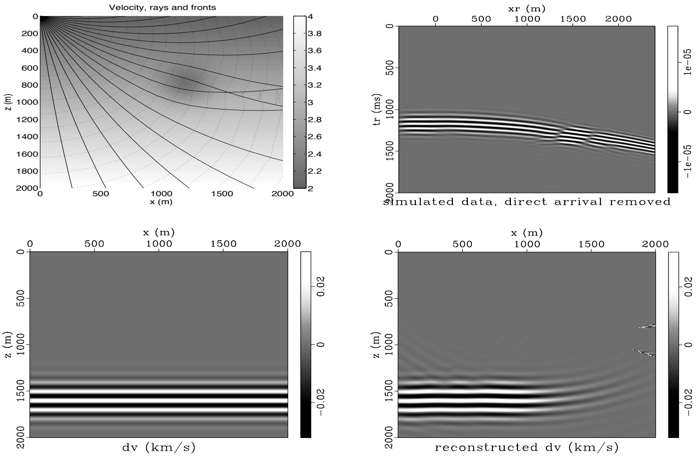
Modified ratio imaging condition derived similarly as above

Example: A vertical gradient medium



Example 1: Velocity perturbation, reconstructed velocity perturbation, and three traces for background medium $v = 2.0 + 0.001z$ (z in meters and v in km/s). Error 8-10 %.

Example: Horizontal reflector with receiver side caustics



Example 2: (a) A velocity model with some rays; (b) Simulated data, with direct arrival removed; (c) Velocity perturbation; (d) Reconstructed velocity perturbation. Error 0-10%

Conclusion and discussion

- We modified reverse time migration to become a linearized inverse scattering method
- We characterized the resolution operator as a partial inverse, a ΨDO with symbol describing aperture effects
- The modified imaging condition suppresses low-frequency artifacts
- Good numerical results in (bandlimited) examples
- Straightforward generalization of the imaging condition to downward propagation methods