Image amplitudes in reverse time migration/inversion

Chris Stolk\textsuperscript{1}, Tim Op ’t Root\textsuperscript{2}, Maarten de Hoop\textsuperscript{3}

\textsuperscript{1} University of Amsterdam
\textsuperscript{2} University of Twente
\textsuperscript{3} Purdue University

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RTM based linearized inversion

- Seismic imaging is mathematically treated as a linearized inverse problem
- Kirchhoff migration can correspond to a method for inverting discontinuities (Beylkin, 1985).
- There are conditions on the background medium. Single source: no multipathing. Multisource: Traveltime Injectivity Condition (Rakesh, 1988; Nolan and Symes 1997; Ten Kroode Et Al., 1998)
- RTM has the advantage that there are no approximations from ray-theory and one-way methods. We will do linearized inverse scattering by a modified reverse time migration (RTM algorithm
The linearized inverse problem

Source field $u_{\text{src}}$

\[
\left( \frac{1}{\nu^2} \partial_t^2 - \nabla_x^2 \right) u_{\text{src}}(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_{\text{src}})\delta(t)
\]

Scattered field: Linearize in the velocity $\frac{1}{\nu(\mathbf{x})^2} \rightarrow \frac{1}{\nu(\mathbf{x})^2}(1 + r(\mathbf{x}))$. $\nu$ the smooth velocity model, $r$ contains singularities

\[
\left( \frac{1}{\nu^2} \partial_t^2 - \nabla_x^2 \right) u_{\text{scat}}(\mathbf{x}, t) = -\frac{r}{\nu^2} \partial_t^2 u_{\text{src}}.
\]

Problem: Determine $r(\mathbf{x})$ from the following data:

$u_{\text{scat}}(\mathbf{x}, t)$ for $\mathbf{x} = (x_1, x_2, x_3)$ in a subset of $x_3 = 0$, $t \in [0, T_{\text{max}}]$
A new view on Reverse Time Migration

- Numerically solve wave equation to obtain
  \[ u_{\text{src}}(x, t) = \text{incoming (source) wave field} \]
  \[ u_{\text{rtc}}(x, t) = \text{reverse time continued receiver field} \]

- New imaging condition

\[
I(x) = \frac{2i}{2\pi\omega^3} \int \frac{\omega^2 \hat{u}_{\text{src}}(x, \omega)\hat{u}_{\text{rtc}}(x, \omega) - v^2 \nabla \hat{u}_{\text{src}}(x, \omega)\nabla \hat{u}_{\text{rtc}}(x, \omega)}{|\hat{u}_{\text{src}}(x, \omega)|^2} d\omega.
\]

(cf. Kiyashchenko et al., 2007)

- Characterization of \( I \)

\[
I(x) = R(x, D_x)r,
\]

Here \( R \) is a pseudodifferential operator that describes the aperture effect, i.e. \( R(x, \xi) = 1 \) for “visible” reflectors (cf. Beylkin, 1985).
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Model problem

Notation $\mathbf{x} = (x_1, x_2, x_3)$.

Source field is plane wave with velocity $(0, 0, \nu)$

$$u_{src}(\mathbf{x}, t) = A \delta(t - \frac{x_3}{\nu})$$

Scattered field slightly simplified

$$\left( \frac{1}{\nu^2} \partial_t^2 - \nabla_x^2 \right) u_{scat}(t, \mathbf{x}) = A \delta(t - \frac{x_3}{\nu}) r(\mathbf{x}).$$

Perfect backpropagation: $u_{rtc}$ solution of a final value problem

$$\left( \frac{1}{\nu^2} \partial_t^2 - \nabla_x^2 \right) u_{rtc}(\mathbf{x}, t) = 0$$

$$u_{rtc}(\mathbf{x}, T_1) = u_{scat}(\mathbf{x}, T_1), \quad \partial_t u_{rtc}(\mathbf{x}, T_1) = \partial_t u_{scat}(\mathbf{x}, T_1), \quad \mathbf{x} \in \mathbb{R}^3.$$
Solving the wave equation

Spatial Fourier transform

\[ (v^{-2} \partial_t^2 + \xi^2) \hat{u}(\xi, t) = \hat{f}(\xi, t). \]

Duhamel’s principle: Let \( \hat{w}(\xi, t, s) \)

\[ (v^{-2} \partial_t^2 + \xi^2) \hat{w}(\xi, t, s, \xi) = 0, \]
\[ \hat{w}(\xi, s, s) = 0 \quad \partial_t \hat{w}(\xi, s, s) = v^2 \hat{f}(s, \xi). \]

Result

\[ \hat{u}(\xi, t) = \int_0^t \left( e^{iv\|\xi\|(t-s)} - e^{-iv\|\xi\|(t-s)} \right) \frac{v^2 \hat{f}(\xi, s)}{2iv\|\xi\|} \, ds \]

Needed: Fourier transform of r.h.s. \( A \delta(t - \frac{x_3}{v}) r(x) \)
Fourier transform \( f(x, t) = A \delta(t - \frac{x_3}{v})r(x) \)

\[
\hat{f}(\xi, t) = \int A \delta(t - \frac{x_3}{v})r(x)e^{-ix\cdot\xi} \, dx
\]

\[
= vA \mathcal{F}_{(x_1, x_2)\mapsto(\xi_1, \xi_2)} r(\xi_1, \xi_2, v_0 t)e^{-i\xi_3 vt}.
\]

Computations involve two similar terms, take only one of those.

\[
\hat{u}_{\text{scat}}(\xi, t) = \int_0^t \frac{v^3}{-2iv\|\xi\|} e^{-iv\|\xi\|(t-s)-i\xi_3 vs} A \mathcal{F}_{(x_1, x_2)\mapsto(\xi_1, \xi_2)} r(\xi_1, \xi_2, vs) \, ds
\]

\[
+ \text{pos. freq. term}
\]

\[
= \frac{Av^2}{-2iv\|\xi\|} e^{-iv\|\xi\|t} \hat{r}(\xi_1, \xi_2, \xi_3 - \|\xi\|) + \text{pos. freq. term},
\]

if \( t \) is after the incoming has passed \( \text{supp}(r) \)

Back to space domain

\[
u_{\text{scat}}(x, t) = 2 \text{Re} \frac{1}{(2\pi)^3} \int \frac{Av^2}{-2iv\|\xi\|} e^{ix\cdot\xi-i\|\xi\|t} \hat{r}(\xi-(0,0,\|\xi\|)) \, d\xi
\]
Interpretation

Scattered field

\[ u_{\text{scat}}(x, t) = 2 \text{Re} \left( \frac{1}{(2\pi)^3} \int \frac{A\nu^2}{-2i\nu\|\xi\|} e^{ix\cdot\xi - i\nu\|\xi\|t} \hat{r}(\xi - (0, 0, \|\xi\|)) \, d\xi \right) \]

Wave vectors

\( \xi \)
\( (0, 0, \|\xi\|) \) outgoing wave number
\( \xi - (0, 0, \|\xi\|) \) incoming wave number
\( \xi - (0, \|\xi\|) \) reflectivity wave number
Modified excitation time imaging condition

Same formula for the backpropagated field! Valid for all $t$

$$u_{rtc}(x, t) = 2 \text{Re} \frac{1}{(2\pi)^3} \int \frac{A v^2}{-2iv\|\xi\|} e^{ix \cdot \xi - iv\|\xi\|t} \hat{r}(\xi - (0, 0, \|\xi\|)) d\xi$$

Basic image: $I_0(x) = u_{rtc}(x, x_3/v)$.

Linearized inverse scattering: Try

$$I(x) = \frac{2}{\sqrt{2}A} (\partial_t + v \partial_{x_3}) u_{rtc}(x, x_3/v)$$

(meaning first take derivatives then insert $(x, t) = (x, x_3/v)$ )

Straightforward calculation yields

$$I(x) = 2 \text{Re} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left(1 - \frac{\xi_3}{\|\xi\|}\right) \hat{r}(\xi - (0, 0, \|\xi\|)) d\xi$$
A change of variables

Copy from previous slide:

\[
I(x) = 2 \text{Re} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( 1 - \frac{\xi_3}{\|\xi\|} \right) \hat{r}(\xi - (0, 0, \|\xi\|)) \, d\xi
\]

Change of variables

\[
\tilde{\xi} = \xi - (0, 0, \|\xi\|)
\]

- Jacobian \(1 - \frac{\xi_3}{\|\xi\|}\)
- Domain \(\tilde{\xi} \in \mathbb{R}^2 \times \mathbb{R}_{<0}\)

Result:

\[
I(x) = 2 \text{Re} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2 \times \mathbb{R}_{<0}} \hat{r}(\tilde{\xi}) e^{ix \cdot \tilde{\xi}} \, d\tilde{\xi} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2 \times \mathbb{R}_{\neq 0}} \hat{r}(\tilde{\xi}) e^{ix \cdot \tilde{\xi}} \, d\tilde{\xi}.
\]

Reconstruction except for \(\xi_3 = 0\), corresponding to reflection over 180 degrees.
Modified ratio imaging condition

We had

\[ I(x) = \frac{2}{\sqrt{2} A} (\partial_t + (0, 0, v) \cdot \nabla_x) u_{rtc}(x, x_3/v) \]

Rewrite into modified ratio imaging condition

- Use that \((0, 0, v) = v^2 \nabla T(x)\), and that

\[ \hat{u}_{src}(x, \omega) = Ae^{-i\omega T(x)}, \]

\[ \nabla_x \hat{u}_{src}(x, \omega) \approx -i\omega \nabla_x T(x)Ae^{-i\omega T(x)}. \]

- Insert factors \(\frac{v^2}{\omega^2}\) left out in model problem

This results in

\[ I(x) = \frac{2i}{2\pi \omega^3} \int \frac{\omega^2 \hat{u}_{src}(x, \omega) \hat{u}_{rtc}(x, \omega) - v^2 \nabla \hat{u}_{src}(x, \omega) \nabla \hat{u}_{rtc}(x, \omega)}{|\hat{u}_{src}(x, \omega)|^2} d\omega. \]
Extension to variable background

Local vs. global analysis
- A local analysis generalizes the explicit formulas to variable coefficients, and leads to explicit inversion formulas.

Tools: microlocal analysis
- Fourier integral operators (FIO’s), pseudodifferential operators (ΨDO’s)
- Description of the mapping properties of operators for localized plane wave components. Work in \((x, \xi)\) or \((x, t, \xi, \omega)\) domain.
Variable coefficients: The wave equation

**Local analysis**
Solve the wave equation using WKB with plane wave initial values. Result is an FIO

\[
    u(y, t) = \frac{1}{(2\pi)^3} \int \int e^{i\alpha(y, t-s, \xi)} a(y, t-s, \xi) \hat{f}(\xi, s) \, ds \, d\xi + \text{pos. freq. term},
\]

\(\alpha\) satisfies the eikonal equation; \(a\) a transport equation. Formula is valid for \(t, s\) in a bounded time interval

**Global analysis**
- Propagation of localized plane wave components along rays, properly described as curves in \((x, t, \xi, \omega)\) space
- Time reversal property is globally valid
Continued scattered field and linearized forward map

Define the continued scattered field \( u_h \), as the “perfect” backpropagated field from a fixed time, full position space

**Local result**
For a localized contribution to \( r \)

\[
u_{h,a}(y, t) = \frac{1}{(2\pi)^3} \int \int e^{i\varphi_T(y, t, x, \xi)} A(y, t, x, \xi) r(x) d\xi d\chi.
\]

With phase function

\[
\varphi_T(y, t, x, \xi) = \alpha(y, t - T(x), \xi) - \xi \cdot x
\]

**Global result**
The map \( F_- : r \mapsto u_h \) is a FIO. Mapping of wave components \((x, \zeta) \mapsto (y, t, \eta, \omega)\): see picture
RT continuation from the boundary

Preprocess data:

- $\Psi DO$ cutoff $\Psi_M(y_1, y_2, t, \eta_1, \eta_2, \omega)$ with three effects
  - Smooth taper near acquisition boundary
  - Remove direct waves
  - Remove tangentially incoming waves

- Normalize wave field to get true amplitude time reversal

Characterize mathematically the aperture effect

**Result** There are $\Psi DO$’s $P_-, P_+$, such that

$$u_{rtc}(x, t) = P_-(t, x, D_x)u_{h,-} + P_+(t, x, D_x)u_{h,+}$$

in which $u_{h,\pm}$ is the part of $u_h$ with $\pm \omega > 0$. To highest order

$$P_\pm(s, x, \xi) = \Psi_M(y_1, y_2, t, \eta_1, \eta_2, \omega)$$

if there is a ray with $\pm \omega > 0$, connecting $(x, s, \xi, \omega)$ with $(y_1, y_2, t, \eta_1, \eta_2, \omega)$
Inverse scattering

Modified excitation time imaging condition

\[ I(x) = \frac{2}{A_{\text{src}}(x)} \partial_t^{-\frac{n+1}{2}} \left[ \partial_t + v^2 \nabla T \cdot \nabla_x \right] u_{\text{rtc}}(x, t) \]

- From the global analysis, there are no nonlocal contributions and
  \[ I(x) = \Psi \text{DO } r(x) \]

- From the local analysis
  \[ I(x) = (R_-(x, D_x) + R_+(x, D_x)) r(x) \]

To highest order

\[ R_\pm(x, \zeta) = P_\pm(x, \xi) \]

if \( \zeta \) and \( \xi \) are related through

\[ \zeta = \xi \pm \omega \nabla T(x). \]

Modified ratio imaging condition derived similarly as above
Example: A vertical gradient medium

Example 1: Velocity perturbation, reconstructed velocity perturbation, and three traces for background medium \( v = 2.0 + 0.001z \) (\( z \) in meters and \( v \) in km/s). Error 8-10 %. 
Example: Horizontal reflector with receiver side caustics

Example 2: (a) A velocity model with some rays; (b) Simulated data, with direct arrival removed; (c) Velocity perturbation; (d) Reconstructed velocity perturbation. Error 0-10%
Conclusion and discussion

- We modified reverse time migration to become a linearized inverse scattering method
- We characterized the resolution operator as a partial inverse, a $\Psi$DO with symbol describing aperture effects
- The modified imaging condition suppresses low-frequency artifacts
- Good numerical results in (bandlimited) examples
- Straightforward generalization of the imaging condition to downward propagation methods